LOW DEGREE EQUATIONS DEFINING THE HILBERT SCHEME

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Abstract. The Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$, parametrizing closed subschemes in the projective space $\mathbb{P}^n$ with Hilbert polynomial $p(t)$ of degree d, is classically constructed as a subscheme of a suitable grassmannian, so that it becomes a closed projective subscheme through the associated Plücker embedding. In this paper we find new global equations for $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ and especially we prove that $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ can be defined scheme-theoretically by homogeneous polynomials of degree smaller than or equal to $d+2$ in the Plücker coordinates. We obtain these equations using a special set of generators of any exterior power of a vector space, that depend linearly on the Plücker coordinates of such vector space in the corresponding grassmannian. Firstly, we exploit these generators to obtain in a simpler way the well-known equations for $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ by Iarrobino and Kleiman and those conjectured by Bayer in 1982 and then proved by Haiman and Sturmfels in 2004. Finally, using combinatorial properties of Borel-fixed ideals and their relations with the geometry of the Hilbert scheme, we obtain our new equations. A procedure for computing this set of equations comes out directly of our proof and it can be fruitfully used in simple cases.

Introduction

The study of the Hilbert scheme is a very active domain in algebraic geometry. The Hilbert scheme $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$, associated to an admissible Hilbert polynomial $p(t)$ and a projective space $\mathbb{P}^n = \operatorname{Proj} k[x_0,\ldots,n_n]$, was firstly introduced by Grothendieck [9] and parametrizes the set of all subschemes of $\mathbb{P}^n$ with Hilbert polynomial $p(t)$.

Let $S = k[x_0,\ldots,x_n]$ be a polynomial ring in $n+1$ variables with coefficients in a field k of characteristic 0 and $q(t) = \binom{t+n}{n} - p(t)$ be the volume polynomial associated to $p(t)$. Classically the Hilbert scheme is constructed as a subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ (see [9, 16]) of the subspaces of dimension $q(r)$ in the base vector space of dimension $\binom{n+r}{n}$ of polynomials of degree r, for an integer r sufficiently large.

Two important results by Gotzmann [7], known as Gotzmann's Regularity Theorem and Gotzmann's Persistence Theorem, give a method to compute the degree r and the condition to be imposed on the points of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ in order to define $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$. The first result allows to compute the minimal suitable degree r directly from the polynomial $p(t)$ and the second theorem ensures that in order for an ideal $I_r \in \mathbf{Gr}^{p(r)}_{S_r}(k)$ to have volume polynomial $q(t)$ it is sufficient to check that $\dim I_{r+1} = q(r+1)$.

Applying these results, in the years many authors dealt with the problem of determining a set of explicit equations defining scheme-theoretically the Hilbert scheme as subscheme of either a grassmannian or a product of two grassmannians or a projective space through the Plücker embedding. There are at least a couple of motivations that from our point of view makes interesting the study of this problem. The first one is the following theoretical question, still open:

can we determine a set of equations that defines ideal-theoretically the Hilbert scheme $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ as subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$?

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The second motivation is more practical and concerns with projecting effective methods for computing a set of equations defining $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ scheme-theoretically as subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$. In fact the number of variables involved in such a computation is very large for the intrinsic complexity of the problem and so it cannot be reduced, therefore we can only try to pull down as much as possible the degree and the number of equations to at least think to do some computation in non-trivial cases.

At the moment the results known give equations of quite large degree, even in simple cases. Iarrobino and Kleiman [12, Proposition C.30] determined a set of equations of degree $q(r+1)+1$ and later Bayer [3] conjectured and then Haiman and Sturmfels [10] proved that there exists a set of equations of degree $n+1$. These two bounds prove to be too large, even in the simplest non-trivial cases. For instance for the Hilbert scheme $\mathbf{Hilb}^{\mathbb{P}^3}_{2}(k)$ of 2 points in $\mathbb{P}^3$, the equations by Iarrobino and Kleiman have degree equal to 19 and those by Bayer, Haiman and Sturmfels degree equal to 4.

More recently Alonso, Brachat and Mourrain [1] showed that for Hilbert schemes of points, a set of equations can be found in degree equal to 2. The basic tool used is represented by border basis, which work only on the case of zero-dimension ideals, so that using this construction no improvement can be deduced for non-constant Hilbert polynomials.

On the other hand, from a local perspective Bertone, Lella and Roggero [4] proved that $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be covered by open subsets defined in local Plücker coordinates (that is in affine open subsets of the grassmannian) by equations of degree smaller than or equal to $d+2$, where d is the degree of the Hilbert polynomial.

These two bounds agree in the case of constant Hilbert polynomials, so we have been encouraged to mix ideas and techniques in order to extend the bound obtained for local equations to global ones.

The main result of our paper is the following:

Theorem 5.3. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ of degree d and Gotzmann number r. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be defined as a closed subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ by equations of degree smaller than or equal to $d+2$.*

Excluding the trivial case of hypersurfaces, this bound is lower than the other two and seems to be more consistent with the object we are studying, because it only depends on the dimension d of the subschemes parametrized by $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$.

Let us now examine the structure of the paper. In Section 2 we discuss properties of the Plücker coordinates. In particular starting from the coordinates of a point $w \in \mathbb{P}(\wedge^q E)$ corresponding to a vector subspace $V = \langle v_1, \ldots, v_q \rangle$ of a vector space E, that is $w = v_1 \wedge \cdots \wedge v_q$, we construct a set of generators of $\wedge^l V$, for any $l = 1, \ldots, q$, depending *linearly* on the Plücker coordinates of w.

Afterward we show that these sets of generators allow us to express in a simpler way the conditions defining the Hilbert scheme as subscheme of the grassmannian. As application of this idea, in Section 3 we give a simple proof of the result by Iarrobino and Kleiman and in Section 4 we obtain in the same simple way the result by Bayer, Haiman and Sturmfels. Indeed both bounds naturally arise applying our construction on the exterior product and the different results only depend on two slightly different strategies.

In Section 5, we begin introducing the last ingredient, on which our proof hard relies, that is combinatorial properties of Borel-fixed ideals and their relation with the geometry of the Hilbert scheme. Then we give a constructive proof of the main result of the paper (Theorem 5.3), first on an algebraically closed ground field of characteristic 0 and in the last section over a local ring (Theorem 6.12).

A procedure for computing this set of global equations comes out directly of the proof, but even if they are much simpler than those introduced by Iarrobino-Kleiman and Bayer-Haiman-Sturmfels, their computation remains hard to achieve except in a few easy cases. Other approaches, for instance as in [14, 4] by a local perspective, appear to be more promising for an effective study of a specific Hilbert scheme.

1. General setting

As already said in the introduction, S will denote the polynomial ring $k[x_0,\dots,x_n]$ with coefficients in a ground field k of characteristic 0. For denoting monomials, we will use the multi-index notation, that is $x_0^{\alpha_0}\cdots x_n^{\alpha_n} = x^\alpha$ for every $\alpha = (\alpha_0,\dots,\alpha_n) \in \mathbb{N}^{n+1}$.

Given a homogeneous ideal $I \subset S$, for any $t \in \mathbb{N}$, I_t is the vector space of the polynomials of degree t in I. $p(t)$ and $q(t)$ will be the Hilbert polynomials of the graded modules S/I and I. Following the notation of [7], we will refer to $p(t)$ as Hilbert polynomial of I and to $q(t)$ as volume polynomial of I. Obviously $p(t)+q(t) = \binom{n+t}{n}$.

Usually r will be the Gotzmann number of the Hilbert polynomial $p(t)$, that is the highest Castelnuovo-Mumford regularity of the sheaf of ideals associated to a subscheme $X \subset \mathbb{P}^n$ with Hilbert polynomial $p(t)$. It can be easily computed applying the Gotzmann's Regularity Theorem [8, Theorem 3.11], indeed writing $p(t)$ in the unique way as

$$p(t) = \binom{t+a_0}{a_0} + \binom{t+a_1-1}{a_1} + \ldots + \binom{t+a_s-s}{a_s}, \qquad a_0 \geqslant a_1 \geqslant \ldots \geqslant a_s,$$

the Gotzmann number r is equal to $s+1$.

Borel-fixed ideals are ideals fixed by the action of the Borel subgroup of the upper triangular matrices of the linear group $\mathrm{GL}(n+1)$. In our assumption on the characteristic of the ground field, they have a strong combinatorial characterization, precisely supposing the variables ordered as $x_n > \ldots > x_0$, a Borel ideal I is a monomial ideal and for any monomial $x^\alpha \in I$, $\frac{x_j}{x_i}x^\alpha$ also belongs to I, for all $j > i$ and $x_i \mid x^\alpha$. For further details see [15, 4, 13].

It is well known that the Castelnuovo-Mumford regularity of the ideal sheaf of a subscheme X in $\mathbb{P}^n$ coincides with the regularity of the saturated ideal $I(X) \subset S$ defining X and that the regularity of a Borel-fixed ideal is equal to the maximal degree of a generator in its monomial basis. So any saturated Borel-fixed ideal I with Hilbert polynomial $p(t)$ is generated in degree smaller than or equal to the Gotzmann number r. In particular the saturated lexsegment ideal associated to the volume polynomial $q(t)$ has regularity equal to r.

In the following, we will denote by $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ the Hilbert scheme parameterizing subschemes of the projective space $\mathbb{P}^n$ with Hilbert polynomial $p(t)$. According to the classical construction of the Hilbert scheme (see [9, 16]), $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be identified with a subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ parametrizing the vector subspaces of dimension $q(r)$ of the vector space S_r. Indeed for any saturated ideal $I \subset S$ with Hilbert polynomial $p(t)$, the eqality $\dim_k I_r = q(r)$ always holds true because the regularity of I is smaller than or equal to the Gotzmann number r. Therefore the key point is to determine the conditions for an ideal $J = (J_r) \subset S$, generated by a vector subspace $J_r \subset S_r$ of dimension $q(r)$, to have volume polynomial $q(t)$. Then we have $J_{r+1} = \langle x_0 J_r,\dots,x_n J_r\rangle$ by definition and $\dim_k J_{r+1} \geqslant q(r+1)$ by Macaulay's Estimate on the Growth of Ideals [8, Theorem 3.3]. Moreover, Gotzmann's Persistence Theorem [8, Theorem 3.8] asserts that the condition $\dim_k J_{r+1} = q(r+1)$ suffices to have that the volume polynomial of J is $q(t)$, so the condition we will impose on the points of the grassmannian is $\dim_k J_{r+1} \leqslant q(r+1)$.

We postpone the discussion of the functorial point of view in Section 6, in which we will consider a local ring instead of a field and consequently the general Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ representing the Hilbert functor $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}$ (see [16, Section 4.3]). However in the following A will

be a finitely generated k-algebra and a local ring with maximal ideal $\mathfrak{m}$ and residue field K. We will denote by T the polynomial ring $A[x_0, \dots, x_n]$ and for any k-vector space V of dimension N, E will be the tensor product $V \otimes_k A$.

2. Plücker coordinates

Definition 2.1. (see [16, Section 4.3.3]) Let V be a k-vector space of finite dimension N and $p \leqslant N$ an integer. Let X be a noetherian scheme over k. The *p-Grassmann functor* of V is the contravariant functor $\mathbf{Gr}_V^p(X)$: (schemes)$^\circ \to$ Sets from the category of (schemes) to the category of Sets which associates to X the set of locally free sheaves of rank p, quotient of the free sheaf $V^* \otimes_k \mathcal{O}_X$ on X

$$\mathbf{Gr}_V^p(X) = \{\mathcal{F} \text{ locally free sheaf of rank } p \mid \ V^* \otimes_k \mathcal{O}_X \to \mathcal{F} \to 0\}\,.$$

This contravariant functor is representable by a scheme called the p-grassmannian of V and that we will denote by $\mathrm{Gr}^p(V)$.

In the case $X = \operatorname{Spec} A$, we have

$$\mathbf{Gr}_V^p(\operatorname{Spec} A) = \{M \text{ free } A\text{-module of rank } p \mid \exists\ F \subset E = V \otimes_k A \text{ s.t. } M = E/F\}\,.$$

In the following we will write $\mathbf{Gr}_V^p(A)$ instead of $\mathbf{Gr}_V^p(\operatorname{Spec} A)$ and we will say that a free A-module $F \subset E$ of rank $q = N - p$ belongs to $\mathbf{Gr}_V^p(A)$ meaning that E/F is a free A-module of rank p.

Let us now describe the grassmannian $\mathbf{Gr}_V^p(A)$. Considered $E/F \in \mathbf{Gr}_V^p(A)$, we have the natural morphism

$$E \ \longrightarrow\ E/F \ \longrightarrow\ 0,$$

and its p-th exterior power:

$$\wedge^p E \ \xrightarrow{\phi}\ \wedge^p(E/F) \ \longrightarrow\ 0.$$

Chosen a basis $\{e_1, \dots, e_N\}$ of E (as a free A-module) and considered the family of all sets of p indices $\mathcal{I} = \{i_1, \dots, i_p\},\ 1 \leqslant i_1 < \dots < i_p \leqslant N$, the Plücker coordinates of E/F are the elements

$$\Delta_{\mathcal{I}} = \phi(e_{i_1}, \dots, e_{i_p}) \in E/F \simeq A, \quad \forall\, \mathcal{I}. \tag{1}$$

This set of coordinates gives the usual projective Plücker embedding:

$$\begin{array}{rccc} \varphi: & \mathbf{Gr}_V^p(A) & \longrightarrow & \mathbb{P}\left(\wedge^p E\right) \\ & E/F & \longmapsto & [\dots : \Delta_{\mathcal{I}}(E/F) : \dots] \end{array} \tag{2}$$

Given a basis $\{\lambda_1, \dots, \lambda_p\}$ of the dual space $F^* = \langle \lambda \in E^* \mid \lambda(F) = 0\rangle$, Plücker coordinates in (1) can be computed through the following determinants:

$$\Delta_{\mathcal{I}} = \begin{vmatrix} \lambda_1(e_{i_1}) & \dots & \lambda_1(e_{i_p}) \\ \vdots & \ddots & \vdots \\ \lambda_p(e_{i_1}) & \dots & \lambda_p(e_{i_p}) \end{vmatrix}. \tag{3}$$

We can also consider another system of coordinates associated to the embedding of the grassmannian $\mathbf{Gr}_V^p(A)$ into the projective space $\mathbb{P}\left(\wedge^q E\right)$:

$$\begin{array}{rccc} \psi: & \mathbf{Gr}_V^p(A) & \longrightarrow & \mathbb{P}\left(\wedge^q E\right) \\ & F = \langle f_1, \dots, f_q\rangle \subset E & \longmapsto & f_1 \wedge \dots \wedge f_q \end{array} \tag{4}$$

where $q = N - p$ and $\{f_1, \dots, f_q\}$ is a basis of $F \in \mathbf{Gr}_V^p(A)$ as a free A-module of rank q. In this case, the Plücker coordinates of $F \in \mathbf{Gr}_V^p(A)$ can be determined by computing the $q \times q$ minors of the $q \times N$ matrix obtained putting on the rows the coefficients of the vectors of any base of F in the canonical basis of E. For every multi-index $\mathcal{K} = \{k_1, \dots, k_q\},\ 1 \leqslant k_1 < \dots < k_q \leqslant N$, we will denote with $\Theta_{\mathcal{K}} \in A$ the Plücker coordinate corresponding to the determinant of the $q \times q$ matrix composed by the columns $k_1, \dots, k_q$, so that $\psi(F) = \sum_{\mathcal{K}} \Theta_{\mathcal{K}}(F)\, e_{k_1} \wedge \dots \wedge e_{k_q}$.

Given any (even not-ordered) set of indices $H = \{h_1, \dots, h_p\}$, we will denote with Δ_H (resp. Θ_H) the determinant of the matrix of the evaluation on the vectors $(e_{h_1}, \dots, e_{h_p})$ (resp. the determinant of the matrix obtained considering the columns $h_1, \dots, h_q$). It is easy to check that $\Delta_H = \varepsilon_H \Delta_{\mathcal{H}}$ (resp. $\Theta_H = \varepsilon_H \Theta_{\mathcal{H}}$), where ε_H is the signature of the permutation σ that orders H, $\mathcal{H} = \sigma(H)$ is the corresponding ordered multi-index and $\Delta_{\mathcal{H}}$ (resp. $\Theta_{\mathcal{H}}$) is a Plücker coordinate. From now on, we will always consider ordered multi-indices.

Given two multi-indices $\mathcal{K} = \{k_1, \dots, k_a\}$ and $\mathcal{H} = \{h_1, \dots, h_b\}$, we will denote by $\mathcal{K}|\mathcal{H}$ the set of indices $\{k_1, \dots, k_a, h_1, \dots, h_b\}$, that in general will not be ordered, whereas we will denote with the union $\mathcal{K} \cup \mathcal{H}$ the ordered multi-index containing the indices belonging to both $\mathcal{K}$ and $\mathcal{H}$. For instance given $\mathcal{K} = \{1, 5\}$ and $\mathcal{H} = \{2\}$, $\mathcal{K}|\mathcal{H} = \{1, 5, 2\}$, $\mathcal{H}|\mathcal{K} = \{2, 1, 5\}$ and $\mathcal{K} \cup \mathcal{H} = \mathcal{H} \cup \mathcal{K} = \{1, 2, 5\}$. Coming back to Plücker coordinates the following relation holds:

$$\Delta_{\mathcal{K}|\mathcal{H}} = \varepsilon_{\mathcal{K}|\mathcal{H}} \Delta_{\mathcal{K}\cup\mathcal{H}} \qquad (\text{resp. } \Theta_{\mathcal{K}|\mathcal{H}} = \varepsilon_{\mathcal{K}|\mathcal{H}} \Theta_{\mathcal{K}\cup\mathcal{H}}).$$

Definition 2.2 ([2])**.** Given a field extension $k \subset K$, an *extensor* of step l in $V(K) = V \otimes_k K$ is an element of $\wedge^l V(K)$ of the form $v_1 \wedge \cdots \wedge v_l$ with $v_1, \dots, v_l$ in $V(K)$.

Given a vector subspace $F \subset V(K)$ of dimension l, we define an extensor associated to F as an element of the form $f_1 \wedge \cdots \wedge f_l \in \wedge^l F$, where $\{f_1, \dots, f_l\}$ is a basis of F. Note that all the extensors associated to F are equal up to multiplication by a non-zero scalar.

In the following, as done in [2], we will identify a vector subspace F of $V(K)$ of dimension l with an extensor of step l associated to it.

We recall Proposition 4.2 of [2].

Proposition 2.3. *Given a field extension $k \subset K$ and the vectors $a_1, \dots, a_p, b_1, \dots, b_q$ in $V(K) = V \otimes_k K$ with $p + q > N$, let us consider the extensors*

$$T = a_1 \wedge \cdots \wedge a_p \qquad \textit{and} \qquad U = b_1 \wedge \cdots \wedge b_q.$$

Let $[\ ] : \big(V(K)\big)^{\times N} \to K$ be any alternating N-linear form, so that we have the following diagram:

$$\begin{tikzcd} \big(V(K)\big)^{\times N} \arrow[rr, "{[\ ]}"] \arrow[dr] & & K \\ & \wedge^N V(K) \arrow[ur] & \end{tikzcd}$$

The following identity holds:

$$\sum_{\substack{\mathcal{H}=\{h_1,\dots,h_{N-q}\} \\ \mathcal{K}=\{k_1,\dots,k_{p+q-N}\} \\ \mathcal{H}\cup\mathcal{K}=\{1,\dots,p\}}} \varepsilon_{\mathcal{H}|\mathcal{K}} [a_{h_1} \wedge \cdots \wedge a_{h_{N-q}}, U]\, a_{k_1} \wedge \cdots \wedge a_{k_{p+q-N}} = \tag{5}$$

$$= \sum_{\substack{\mathcal{I}=\{i_1,\dots,i_{p+q-N}\} \\ \mathcal{J}=\{j_1,\dots,j_{N-q}\} \\ \mathcal{I}\cup\mathcal{J}=\{1,\dots,q\}}} \varepsilon_{\mathcal{I}|\mathcal{J}} [T, b_{j_1} \wedge \cdots \wedge b_{j_{N-q}}]\, b_{i_1} \wedge \cdots \wedge b_{i_{p+q-N}} \tag{6}$$

We denote these two terms by $T * U$ and we will call $*$ the meet operator *associated to $[\ ]$.*

Proposition 2.4 ([2])**.** *Let $k \subset K$ be a field and T and U be two extensors associated to two vector subspaces $F \in \mathbf{Gr}^p_V(K)$ and $G \in \mathbf{Gr}^q_V(K)$ ($F \subset V \otimes_k K$ and $G \subset V \otimes_k K$) such that $p + q \geqslant N$ (i.e $F \cup G$ spans the vector space $V(K) = V \otimes_k K$ and $F \cap G \neq 0$). Let $*$ be the meet operator associated to any N-linear form $[\ ]$ which consists of a basis of $\big(\wedge^N V(K)\big)^* \simeq K$. Then $T * U$ is an extensor associated to the vector subspace $F \cap G$ of dimension $p + q - N$.*

Proof. See [2, Proposition 4.3]. ∎

Proposition 2.5. *Let $F \in \mathbf{Gr}_V^p(A)$ be a free submodule of E, $q = N - p$ and $1 \leqslant m \leqslant q$. Let $[\ldots : \Delta_{\mathcal{I}}(F) : \ldots]$ be its coordinates with respect to the Plücker embedding given in (2), then the elements*

$$\delta_{\mathcal{J}}^m(F) = \sum_{\substack{\mathcal{I}=\{i_1,\ldots,i_p\}\\ \mathcal{H}=\{h_1,\ldots,h_m\}\\ \mathcal{H}\cup\mathcal{I}=\mathcal{J}}} \varepsilon_{\mathcal{H}|\mathcal{I}}\, \Delta_{\mathcal{I}}(F)\, e_{h_1} \wedge \cdots \wedge e_{h_m}, \quad \forall\ \mathcal{J} = \{j_1,\ldots,j_{p+m}\}, \tag{7}$$

generate $\wedge^m F$.

Proof. Tensoring by the residue field and using Nakayama's Lemma, we can assume without loss of generality that $A = K$ is a field.

Let us use equation (6) with $T = F \in \mathbf{Gr}_V^p(K)$ and $U = B_{\mathcal{J}} = e_{j_1} \wedge \cdots \wedge e_{j_{p+m}} \in \mathbf{Gr}_V^{N-p-m}(K)$. By Proposition 2.4, we have that

$$F * B_{\mathcal{J}} = \sum_{\substack{\mathcal{I}=\{i_1,\ldots,i_p\}\\ \mathcal{H}=\{h_1,\ldots,h_m\}\\ \mathcal{H}\cup\mathcal{I}=\mathcal{J}}} \varepsilon_{\mathcal{H}|\mathcal{I}}\, [F, B_{\mathcal{I}}]\, B_{\mathcal{H}}$$

is an extensor associated to $F \cap \langle B_{\mathcal{J}}\rangle$ in $V(K)$. If we take the N-linear form $[\ ]$ equal to the determinant in a basis of E which consists of the union of a basis of F and a basis of E/F, then $[F, B_{\mathcal{I}}] = \Delta_{\mathcal{I}}(F)$, therefore

$$F * B_{\mathcal{J}} = \sum_{\substack{\mathcal{I}=\{i_1,\ldots,i_p\}\\ \mathcal{H}=\{h_1,\ldots,h_m\}\\ \mathcal{H}\cup\mathcal{I}=\mathcal{J}}} \varepsilon_{\mathcal{H}|\mathcal{I}}\, \Delta_{\mathcal{I}}(F)\, B_{\mathcal{H}} = \delta_{\mathcal{J}}^m(F)$$

generates $\wedge^m\big(F \cap \langle B_{\mathcal{J}}\rangle\big)$. Thus the family

$$\delta_{\mathcal{J}}^m(F) = F * B_{\mathcal{J}}, \qquad \forall\ \mathcal{J} = \{j_1,\ldots,j_{p+m}\}$$

generates

$$\bigcup_{\mathcal{J}} \Big(\wedge^m \big(F \cap \langle B_{\mathcal{J}}\rangle\big)\Big) = \wedge^m F. \qquad \square$$

Proposition 2.6. *Let $F \in \mathbf{Gr}_V^p(A)$ be a free submodule of E, $q = N - p$ and $1 \leqslant m \leqslant q$. Let $[\ldots : \Theta_{\mathcal{K}}(F) : \ldots]$ be its coordinates with respect to the Plücker embedding given in (4), then the family of elements*

$$\theta_{\mathcal{J}}^m(F) = \sum_{\mathcal{H}=\{h_1,\ldots,h_m\}} \varepsilon_{\mathcal{J}|\mathcal{H}}\, \Theta_{\mathcal{J}\cup\mathcal{H}}(F)\, e_{h_1} \wedge \cdots \wedge e_{h_m}, \qquad \forall\ \mathcal{J} = \{j_1,\ldots,j_{q-m}\}, \tag{8}$$

generate $\wedge^m F$.

Proof. As done in the proof of Proposition 2.5, tensoring by the residue field K of A and using Nakayama's Lemma, we can assume $A = K$.

Let $\mathcal{I}$ be equal to the multi-index such that $\mathcal{J}\cup\mathcal{I} = \{1,\ldots,N\}$. Let us apply the meet operator between $T = F = \sum_{\mathcal{K}} \Theta_{\mathcal{K}}(F)\, e_{k_1} \wedge \cdots \wedge e_{k_q} \in \wedge^q V(K)$ and $U = B_{\mathcal{I}} = e_{i_1} \wedge \cdots \wedge e_{i_{N-q+m}} \in \mathbf{Gr}_V^{q-m}(K)$ according to (5)

$$\begin{aligned} F * B_{\mathcal{I}} &= \sum_{\mathcal{K}} \Theta_{\mathcal{K}}(F)\, e_{k_1} \wedge \cdots \wedge e_{k_q} * B_{\mathcal{I}} = \\ &= \sum_{\substack{\mathcal{K}\\ \mathcal{K}=\mathcal{J}\cup\mathcal{H}}} \Theta_{\mathcal{K}}(F)\, \varepsilon_{\mathcal{J}|\mathcal{H}}\, e_{h_1} \wedge \cdots \wedge e_{h_m} [e_{j_1},\ldots,e_{j_{q-m}}, e_{i_1},\ldots,e_{i_{N-q+m}}]. \end{aligned}$$

If we consider a N-linear form $[\ ]$ such that $[e_{j_1},\dots,e_{j_{q-m}},e_{i_1},\dots,e_{i_{N-q+m}}]$ is equal to 1, we obtain

$$F * B_{\mathcal{I}} = \sum_{\substack{\mathcal{K} \\ \mathcal{K}=\mathcal{J}\cup\mathcal{H}}} \Theta_{\mathcal{K}}(F)\,\varepsilon_{\mathcal{J}|\mathcal{H}}\, e_{h_1}\wedge\cdots\wedge e_{h_m} = \sum_{\mathcal{H}} \varepsilon_{\mathcal{J}|\mathcal{H}}\,\Theta_{\mathcal{J}\cup\mathcal{H}}(F)\, e_{h_1}\wedge\cdots\wedge e_{h_m} = \theta^m_{\mathcal{J}}(F).$$

Hence $\theta^m_{\mathcal{J}}(F)$ is an extensor associated to $F\cap\langle B_{\mathcal{I}}\rangle$ in $V(K)$ (by Proposition 2.4) and generates $\wedge^m\big(F\cap\langle B_{\mathcal{I}}\rangle\big)$. Thus the family $\theta^m_{\mathcal{J}}(F)$, $\forall\ \mathcal{J}=\{j_1,\dots,j_{q-m}\}$ generates

$$\bigcup_{\mathcal{J}}\Big(\wedge^m\left(F\cap\langle B_{\mathcal{I}}\rangle\right)\Big) = \wedge^m F. \qquad \square$$

Example 2.7. Let us apply these results to the grassmannian $\mathbf{Gr}^2_{k^6}(k)$. Let $\{e_1,\dots,e_6\}$ be the fixed basis of the vector space $E = V\otimes_k k = V = k^6$ and let us consider the vector subspace $F = \langle f_1 = e_1, f_2 = e_2, f_3 = e_3 + 2e_4, f_4 = e_5 - e_6\rangle$. A basis for the quotient space E/F is $\{e_4, e_6\}$ and a basis of F^* is $\{\lambda_1 = e_5^* + e_6^*, \lambda_2 = 2e_3^* - e_4^*\}$.

Applying the method showed in (3), we obtain that

$$\Delta_{35}(F) = -2,\quad \Delta_{36}(F) = -2,\quad \Delta_{45}(F) = 1,\quad \Delta_{46}(F) = 1$$

and the other 11 Plücker coordinates vanish.

Using the other system of coordinates, the 4 coordinates that do not vanish are

$$\Theta_{1235}(F) = 1,\quad \Theta_{1236}(F) = -1,\quad \Theta_{1245}(F) = 2,\quad \Theta_{1246}(F) = -2.$$

Let us now apply Proposition 2.5 and Proposition 2.6 to compute two systems of generators of $\wedge^2 F$.

- By Proposition 2.5,

$$\begin{aligned}
&\delta^2_{1235}(F) = \Delta_{35}(F)e_1\wedge e_2 = \mathbf{-2e_1\wedge e_2} = \delta^2_{1236}(F) = -2\delta^2_{1245}(F) = -2\delta^2_{1246}(F),\\
&\delta^2_{1345}(F) = \Delta_{45}(F)e_1\wedge e_3 - \Delta_{35}(F)e_1\wedge e_4 = \mathbf{e_1\wedge e_3 + 2e_1\wedge e_4} = \delta^2_{1346}(F),\\
&\delta^2_{1345}(F) = -\Delta_{36}(F)e_1\wedge e_5 + \Delta_{35}(F)e_1\wedge e_6 = \mathbf{2e_1\wedge e_5 - 2e_1\wedge e_6} = -2\delta^2_{1456}(F),\\
&\delta^2_{2345}(F) = \Delta_{45}(F)e_2\wedge e_3 - \Delta_{35}(F)e_2\wedge e_4 = \mathbf{e_2\wedge e_3 + 2e_2\wedge e_4} = \delta^2_{2346}(F),\\
&\delta^2_{2356}(F) = -\Delta_{36}(F)e_2\wedge e_5 + \Delta_{35}(F)e_2\wedge e_6 = \mathbf{2e_2\wedge e_5 - 2e_2\wedge e_6} = -2\delta^2_{2456}(F),\\
&\delta^2_{3456}(F) = -\Delta_{46}(F)e_3\wedge e_5 + \Delta_{45}(F)e_3\wedge e_6 + \Delta_{36}(F)e_4\wedge e_5 - \Delta_{35}(F)e_4\wedge e_6 =\\
&\qquad\qquad = \mathbf{-e_3\wedge e_5 + e_3\wedge e_6 - 2e_4\wedge e_5 + 2e_4\wedge e_6}
\end{aligned}$$

- By Proposition 2.6,

$$\begin{aligned}
&\theta^2_{12}(F) = \Theta_{1235}(F)e_3\wedge e_5 + \Theta_{1236}(F)e_3\wedge e_6 + \Theta_{1245}(F)e_4\wedge e_5 + \Theta_{1246}(F)e_4\wedge e_6 =\\
&\qquad\qquad = \mathbf{e_3\wedge e_5 - e_3\wedge e_6 + 2e_4\wedge e_5 - 2e_4\wedge e_6},\\
&\theta^2_{13}(F) = -\Theta_{1235}(F)e_2\wedge e_5 - \Theta_{1236}(F)e_2\wedge e_6 = \mathbf{-e_2\wedge e_5 + e_2\wedge e_6} = \frac{\theta^2_{14}(F)}{2},\\
&\theta^2_{15}(F) = \Theta_{1235}(F)e_2\wedge e_3 + \Theta_{1245}(F)e_2\wedge e_4 = \mathbf{e_2\wedge e_3 + 2e_2\wedge e_4} = -\theta^2_{16}(F),\\
&\theta^2_{23}(F) = \Theta_{1235}(F)e_1\wedge e_5 + \Theta_{1236}(F)e_1\wedge e_6 = \mathbf{e_1\wedge e_5 - e_1\wedge e_6} = \frac{\theta^2_{24}(F)}{2},\\
&\theta^2_{25}(F) = -\Theta_{1235}(F)e_1\wedge e_3 - \Theta_{1245}(F)e_1\wedge e_4 = \mathbf{-e_1\wedge e_3 - 2e_1\wedge e_4} = -\theta^2_{26}(F),\\
&\theta^2_{35}(F) = \Theta_{1235}(F)e_1\wedge e_2 = \mathbf{e_1\wedge e_2} = -\theta^2_{36}(F) = \frac{\theta^2_{45}(F)}{2} = -\frac{\theta^2_{46}(F)}{2}
\end{aligned}$$

We can see that any generator arising with the first method appears also in the second list (at most multiplied by a scalar), so the two spaces generated by these two families of vectors in $\wedge^2 E$ are the same and equal to $\wedge^2 F$.

Remark 2.8. It can be easily proved that the two systems of coordinates $[\ldots : \Delta_{\mathcal{I}} : \ldots]$ and $[\ldots : \Theta_{\mathcal{K}} : \ldots]$ are equivalent, indeed for every multi-index $\mathcal{K}$ of q indices,

$$\Theta_{\mathcal{K}} = -\varepsilon_{\mathcal{K}|\mathcal{I}}\,\Delta_{\mathcal{I}},$$

where $\mathcal{K} \cup \mathcal{I} = \{1, \ldots, N\}$ and $\varepsilon_{\mathcal{K}|\mathcal{I}}$ is the signature of the permutation σ that orders $\mathcal{K}|\mathcal{I}$, that is $\sigma(\mathcal{K}|\mathcal{I}) = \mathcal{K} \cup \mathcal{I}$. An analogous relation can be extended to the generators $\left\{\delta^m_{\mathcal{J}}(F)\right\}$ and $\{\theta^m_{\mathcal{I}}(F)\}$ of $\wedge^m F$; indeed if we consider $\mathcal{J} = \{j_1, \ldots, j_{p+m}\}$ and $\mathcal{I} = \{i_1, \ldots, i_{q-m}\}$ such that $\mathcal{J} \cup \mathcal{I} = \{1, \ldots, N\}$,

$$\delta^m_{\mathcal{J}}(F) = -\varepsilon_{\mathcal{J}|\mathcal{I}}\,\theta^m_{\mathcal{I}}(F),$$

where as before, $\varepsilon_{\mathcal{J}|\mathcal{I}}$ is the signature of the permutation that orders $\mathcal{J}|\mathcal{I}$.

For this reason, from now on we will only use the set of Plücker coordinates $[\ldots : \Delta_{\mathcal{I}} : \ldots]$, describing the embedding of the Grassmannian $\mathbf{Gr}^p_V(A)$ in the projective space $\mathbb{P}(\wedge^p E)$.

Definition 2.9. In the following we will consider as base vector space $V = S_r$, the homogeneous polynomials of degree r in S, with its standard monomial basis. Given a Hilbert polynomial $p(t)$ with Gotzmann number equal to r, we are interested in subspaces of S_r (more generally $T_r = S_r \otimes_k A$) of dimension $q(r)$, that is elements of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ (more generally $\mathbf{Gr}^{p(r)}_{S_r}(A)$). Given an ideal $I \subset S$ (resp. $I \subset T$), we will write that $I \in \mathbf{Gr}^{p(r)}_{S_r}(k)$ (resp. $\mathbf{Gr}^{p(r)}_{S_r}(A)$) meaning that the module S_r/I_r (resp. T_r/I_r) is free of rank $p(r)$ and that $I \in \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ (resp. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(A)$) meaning that the module S_t/I_t (resp. T_t/I_t) is free of rank $p(t)$, $\forall\ t \geqslant r$.

For every m, elements of the type $\delta^m_{\mathcal{J}}$ become

$$\delta^m_{\mathcal{J}} = \sum_{\substack{\mathcal{I}=\{i_1,\ldots,i_p\}\\ \mathcal{H}=\{h_1,\ldots,h_m\}\\ \mathcal{H}\cup\mathcal{I}=\mathcal{J}}} \varepsilon_{\mathcal{H}|\mathcal{I}}\,\Delta_{\mathcal{I}}\, x^{\alpha(h_1)} \wedge \cdots \wedge x^{\alpha(h_m)},$$

where

$$\alpha : \left\{1, \ldots, \binom{n+r}{n}\right\} \quad \longrightarrow \quad \left\{(a_0, \ldots, a_n) \in \mathbb{N}^{n+1} \;\middle|\; \sum_{i=0}^{n} a_i = r\right\}$$

is a bijection from the integers, from 1 up to $\binom{n+r}{n}$, to the multi-indices defining monomials of degree r such that

$$x^{\alpha(1)} >_{\texttt{DegRevLex}} x^{\alpha(2)} >_{\texttt{DegRevLex}} \cdots >_{\texttt{DegRevLex}} x^{\alpha(\binom{n+r}{n})}.$$

Moreover we define

- $\mathcal{B}_m = \left\{\delta^m_{\mathcal{J}},\ \forall\ \mathcal{J} = \{j_1, \ldots, j_{p(r)+m}\}\right\}$,
- $$x_i\,\delta^m_{\mathcal{J}} = \sum_{\substack{\mathcal{I}=\{i_1,\ldots,i_p\}\\ \mathcal{H}=\{h_1,\ldots,h_m\}\\ \mathcal{H}\cup\mathcal{I}=\mathcal{J}}} \varepsilon_{\mathcal{H}|\mathcal{I}}\,\Delta_{\mathcal{I}} \left(x_i x^{\alpha(h_1)}\right) \wedge \cdots \wedge \left(x_i x^{\alpha(h_m)}\right),$$
- $x_i\mathcal{B}_m = \left\{x_i\,\delta^m_{\mathcal{J}},\ \forall\ \mathcal{J} = \{j_1, \ldots, j_{p(r)+m}\}\right\}$.

3. Iarrobino-Kleiman equations

Theorem 3.1 (Iarrobino, Kleiman [12]). *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number r. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be defined as a closed subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ by equations of degree $q(r+1)+1$.*

Proof. Let us fix the usual monomial basis on S_r and the associated Plücker coordinates for $\mathbf{Gr}^{p(r)}_{S_r}(k)$ in accord with Definition 2.9 and let $q(t) = \binom{n+t}{n} - p(t)$. Moreover let I be the ideal generated in degree r by the polynomials representing the elements $\delta^1_{\mathcal{J}}$:

$$I = (\mathcal{B}_1).$$

By Proposition 2.5, we know that $\dim I_r = q(r)$, so to obtain equations for $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ it suffices to impose $\dim I_{r+1} \leqslant q(r+1)$. Let us consider the matrix with $\binom{n+r+1}{n}$ columns corresponding to the monomial basis of S_{r+1} and $|\mathcal{B}_1| \cdot (n+1)$ rows corresponding to the polynomials contained in $x_i\mathcal{B}_1$, $\forall\, i = 0, \ldots, n$. The rows of this matrix span $x_0 I_r + \ldots + x_n I_r = I_{r+1}$, so we obtain the equation of $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ by imposing that all the minors of dimension $q(r+1)+1$ vanish. $\square$

Example 3.2. Let us now see how to compute the Iarrobino-Kleiman equations for $\mathbf{Hilb}^{\mathbb{P}^2}_2(K)$ with a field extension $k \subset K$. Since the Gotzmann number is 2, we have to consider the grassmannian $\mathbf{Gr}^2_{k[x,y,z]_2}(K)$, with $k[x,y,z]_2 \simeq k^6$ (as in Example 2.7) with monomial basis $\{e_1 = x^2, e_2 = xy, e_3 = y^2, e_4 = xz, e_5 = yz, e_6 = z^2\}$. Following the proof of Theorem 3.1, we consider the ideal $I = (\mathcal{B}_1)$ and we impose that any subset of $q(3)+1 = 9$ polynomials of $\{x\mathcal{B}_1, y\mathcal{B}_1, z\mathcal{B}_1\}$ is dependent. For instance, considering the polynomials represented in the following matrix

$$\begin{array}{c|cccccccccc|}
 & x^3 & x^2y & xy^2 & y^3 & x^2z & xyz & y^2z & xz^2 & yz^2 & z^3 \\
\hline
x\,\delta^1_{126} & \Delta_{26} & -\Delta_{16} & 0 & 0 & 0 & 0 & 0 & \Delta_{12} & 0 & 0 \\
x\,\delta^1_{156} & \Delta_{56} & 0 & 0 & 0 & 0 & -\Delta_{16} & 0 & \Delta_{15} & 0 & 0 \\
x\,\delta^1_{234} & 0 & \Delta_{34} & -\Delta_{24} & 0 & \Delta_{23} & 0 & 0 & 0 & 0 & 0 \\
x\,\delta^1_{356} & 0 & 0 & \Delta_{56} & 0 & 0 & -\Delta_{36} & 0 & \Delta_{35} & 0 & 0 \\
y\,\delta^1_{123} & 0 & \Delta_{23} & -\Delta_{13} & \Delta_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
y\,\delta^1_{345} & 0 & 0 & 0 & \Delta_{45} & 0 & -\Delta_{35} & \Delta_{34} & 0 & 0 & 0 \\
z\,\delta^1_{146} & 0 & 0 & 0 & 0 & \Delta_{46} & 0 & 0 & -\Delta_{16} & 0 & \Delta_{14} \\
z\,\delta^1_{234} & 0 & 0 & 0 & 0 & 0 & \Delta_{34} & -\Delta_{24} & \Delta_{23} & 0 & 0 \\
z\,\delta^1_{456} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{56} & -\Delta_{46} & \Delta_{45} \\
\hline
\end{array} \tag{9}$$

the dependency condition corresponds to the vanishing of the minors of order 9:

- $-\Delta_{16}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{35}\Delta_{45}\Delta_{46}^2 + \Delta_{13}\Delta_{16}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{45}\Delta_{46}^2 + \Delta_{15}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{36}\Delta_{45}\Delta_{46}^2 +$
  $-\Delta_{13}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{36}\Delta_{45}\Delta_{46}^2 - \Delta_{16}^2\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{13}\Delta_{16}^2\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56} +$
  $-\Delta_{12}\Delta_{16}\Delta_{23}\Delta_{26}\Delta_{34}^2\Delta_{46}^2\Delta_{56} - \Delta_{12}\Delta_{15}\Delta_{26}\Delta_{34}^3\Delta_{46}^2\Delta_{56} + \Delta_{12}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{46}^2\Delta_{56} +$
  $+\Delta_{12}\Delta_{16}\Delta_{24}\Delta_{34}^2\Delta_{35}\Delta_{46}^2\Delta_{56} - \Delta_{12}\Delta_{16}\Delta_{24}^2\Delta_{35}^2\Delta_{46}^2\Delta_{56} + \Delta_{12}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{46}^2\Delta_{56} +$
  $-\Delta_{12}\Delta_{23}\Delta_{24}^2\Delta_{36}\Delta_{45}\Delta_{46}^2\Delta_{56} + \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{45}\Delta_{46}^2\Delta_{56} + \Delta_{12}\Delta_{16}^2\Delta_{23}\Delta_{34}^2\Delta_{46}\Delta_{56}^2 +$
  $-\Delta_{12}\Delta_{16}^2\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{46}\Delta_{56}^2 + \Delta_{12}^2\Delta_{34}^3\Delta_{46}^2\Delta_{56}^2 - \Delta_{12}^2\Delta_{24}\Delta_{34}\Delta_{35}\Delta_{46}^2\Delta_{56}^2,$
- $\Delta_{16}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{35}\Delta_{45}^2\Delta_{46} - \Delta_{13}\Delta_{16}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{45}^2\Delta_{46} - \Delta_{15}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{36}\Delta_{45}^2\Delta_{46} +$
  $+\Delta_{13}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{36}\Delta_{45}^2\Delta_{46} + \Delta_{16}^2\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{45}^2\Delta_{56} - \Delta_{13}\Delta_{16}^2\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}^2\Delta_{56} +$
  $+\Delta_{12}\Delta_{16}\Delta_{23}\Delta_{26}\Delta_{34}^2\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{12}\Delta_{15}\Delta_{26}\Delta_{34}^3\Delta_{45}\Delta_{46}\Delta_{56} - \Delta_{12}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{45}\Delta_{46}\Delta_{56} +$
  $-\Delta_{12}\Delta_{16}\Delta_{24}\Delta_{34}^2\Delta_{35}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{12}\Delta_{16}\Delta_{24}^2\Delta_{35}^2\Delta_{45}\Delta_{46}\Delta_{56} - \Delta_{12}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56} +$
  $+\Delta_{12}\Delta_{23}\Delta_{24}^2\Delta_{36}\Delta_{45}^2\Delta_{46}\Delta_{56} - \Delta_{12}\Delta_{13}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{45}^2\Delta_{46}\Delta_{56} + \Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{45}\Delta_{56}^2 +$
  $-\Delta_{12}\Delta_{16}^2\Delta_{23}\Delta_{34}^2\Delta_{45}\Delta_{56}^2 + \Delta_{12}\Delta_{16}^2\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{45}\Delta_{56}^2 - \Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{56}^2 +$
  $-\Delta_{12}^2\Delta_{34}^3\Delta_{45}\Delta_{46}\Delta_{56}^2 + \Delta_{12}^2\Delta_{24}\Delta_{34}\Delta_{35}\Delta_{45}\Delta_{46}\Delta_{56}^2 - \Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{34}^2\Delta_{56}^3 +$
  $+\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{56}^3,$
- $-\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{34}^2\Delta_{46}\Delta_{56}^2 +$
  $-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{46}\Delta_{56}^2,$
- $\Delta_{14}\Delta_{16}\Delta_{23}^3\Delta_{26}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{14}\Delta_{15}\Delta_{23}^2\Delta_{26}\Delta_{34}\Delta_{45}\Delta_{46}\Delta_{56} - \Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{34}\Delta_{35}\Delta_{45}\Delta_{46}\Delta_{56} +$
  $-\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56} + \Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{35}\Delta_{46}\Delta_{56}^2 - \Delta_{12}\Delta_{14}\Delta_{23}^2\Delta_{34}\Delta_{45}\Delta_{46}\Delta_{56}^2,$

- $-\Delta_{14}\Delta_{15}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{45}\Delta_{46}\Delta_{56}+\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{45}\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{34}\Delta_{46}\Delta_{56}^2+$
$+\Delta_{12}\Delta_{14}\Delta_{23}^2\Delta_{24}\Delta_{45}\Delta_{46}\Delta_{56}^2$,
- $-\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{35}\Delta_{45}\Delta_{46}+\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{45}\Delta_{46}+\Delta_{14}\Delta_{15}\Delta_{23}\Delta_{24}^2\Delta_{26}\Delta_{36}\Delta_{45}\Delta_{46}+$
$-\Delta_{13}\Delta_{14}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{36}\Delta_{45}\Delta_{46}-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{26}\Delta_{34}^2\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{26}\Delta_{34}^3\Delta_{46}\Delta_{56}+$
$+\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{24}\Delta_{26}\Delta_{34}\Delta_{35}\Delta_{46}\Delta_{56}+\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{24}\Delta_{34}^2\Delta_{35}\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{24}^2\Delta_{35}^2\Delta_{46}\Delta_{56}+$
$+\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{23}\Delta_{24}^2\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56}+\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{24}\Delta_{34}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56}+$
$+\Delta_{12}^2\Delta_{14}\Delta_{34}^3\Delta_{46}\Delta_{56}^2-\Delta_{12}^2\Delta_{14}\Delta_{24}\Delta_{34}\Delta_{35}\Delta_{46}\Delta_{56}^2$,
- $\Delta_{14}\Delta_{16}\Delta_{23}^3\Delta_{26}\Delta_{34}\Delta_{46}\Delta_{56}+\Delta_{14}\Delta_{15}\Delta_{23}^2\Delta_{26}\Delta_{34}^2\Delta_{46}\Delta_{56}-\Delta_{14}\Delta_{15}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{35}\Delta_{46}\Delta_{56}+$
$-\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{34}^2\Delta_{35}\Delta_{46}\Delta_{56}+\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}^2\Delta_{46}\Delta_{56}-\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{34}\Delta_{36}\Delta_{46}\Delta_{56}+$
$-\Delta_{12}\Delta_{14}\Delta_{23}^2\Delta_{34}^2\Delta_{46}\Delta_{56}^2+\Delta_{12}\Delta_{14}\Delta_{23}^2\Delta_{24}\Delta_{35}\Delta_{46}\Delta_{56}^2$,
- $-\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{35}\Delta_{45}\Delta_{46}+\Delta_{14}\Delta_{15}\Delta_{23}^2\Delta_{24}\Delta_{26}\Delta_{36}\Delta_{45}\Delta_{46}+\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{34}^2\Delta_{35}\Delta_{46}\Delta_{56}+$
$-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}^2\Delta_{46}\Delta_{56}+\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{34}\Delta_{36}\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{23}^2\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56}$,
- $\Delta_{13}\Delta_{14}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{26}\Delta_{35}\Delta_{45}\Delta_{46}-\Delta_{13}\Delta_{14}\Delta_{15}\Delta_{23}\Delta_{24}\Delta_{26}\Delta_{36}\Delta_{45}\Delta_{46}-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{26}\Delta_{34}\Delta_{46}\Delta_{56}+$
$-\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{23}\Delta_{26}\Delta_{34}^2\Delta_{46}\Delta_{56}+\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{23}\Delta_{24}\Delta_{26}\Delta_{35}\Delta_{46}\Delta_{56}+\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{46}\Delta_{56}+$
$+\Delta_{12}^2\Delta_{14}\Delta_{23}\Delta_{34}^2\Delta_{46}\Delta_{56}^2-\Delta_{12}^2\Delta_{14}\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{46}\Delta_{56}^2$,
- $-\Delta_{13}\Delta_{14}\Delta_{16}^2\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{45}\Delta_{46}+\Delta_{13}\Delta_{14}\Delta_{15}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{36}\Delta_{45}\Delta_{46}+\Delta_{12}\Delta_{14}\Delta_{16}^2\Delta_{23}^2\Delta_{34}\Delta_{46}\Delta_{56}+$
$+\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{16}\Delta_{23}\Delta_{34}^2\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{15}\Delta_{16}\Delta_{23}\Delta_{24}\Delta_{35}\Delta_{46}\Delta_{56}-\Delta_{12}\Delta_{14}\Delta_{16}\Delta_{23}^2\Delta_{24}\Delta_{45}\Delta_{46}\Delta_{56}$.

4. Bayer equations

To reduce the degree of equations and to reach the bound given by Bayer in [3], we collect together rows in the matrix of polynomials in degree $r+1$ that come from polynomials in degree r multiplied by the same variable and to express the minors of such submatrices as *linear* combinations of Plücker coordinates. For instance in Example 3.2, we may express the minors of the matrix in (9), putting together the first 4 rows (representing 4 generators multiplied by x), the fifth and the sixth (corresponding to 2 generators multiplied by y) and the last 3 (3 generators multiplied by z). In this way we obtain equations of degree 3.

Theorem 4.1 (Bayer [3])**.** *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ with Gotzmann number r. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be defined as a closed subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ by equations of degree $n+1$.*

Proof. Let us consider the point $I=\langle\mathcal{B}_1\rangle$ of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ and let $q(t)=\binom{n+t}{n}-p(t)$. The condition on the dimension of I_{r+1}, that we impose to obtain equations of $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$, can be rewritten as

$$\dim_k I_{r+1}\leqslant q(r+1)\quad\Longleftrightarrow\quad\wedge^{q(r+1)+1}I_{r+1}=0. \tag{10}$$

In Theorem 3.1, we look at $\wedge^{q(r+1)+1}I_{r+1}$ as

$$\wedge^{q(r+1)+1}I_{r+1}=\left\langle\bigwedge_{i=1}^{q(r+1)+1}l_i\,\delta^1_{\mathcal{J}_i}\ \middle|\ \begin{array}{l}\forall\ l_i\in\{x_0,\dots,x_n\}\\ \forall\text{ multi-index }\mathcal{J}_i\text{ of size }p(r)+1\end{array}\right\rangle.$$

But to compute $\wedge^{q(r+1)+1}I_{r+1}$, we can take advantage of the generators of $\wedge^m I_{r+1}$ for every m, indeed

$$\wedge^{q(r+1)+1}I_{r+1}=\left\langle x_0\,\delta^{m_0}_{\mathcal{J}_0}\wedge\cdots\wedge x_n\,\delta^{m_n}_{\mathcal{J}_n}\ \middle|\ \begin{array}{l}\forall\ x_0\delta^{m_0}_{\mathcal{J}_0}\in x_0\mathcal{B}_{m_0},\dots,x_n\delta^{m_n}_{\mathcal{J}_n}\in x_n\mathcal{B}_{m_n}\\ \forall\ m_i\leqslant q(r)\text{ s.t. }\sum_i m_i=q(r+1)+1\end{array}\right\rangle.$$

Every generator of this type has coefficients represented by homogeneous polynomials in the Plücker coordinates of degree $n+1$. By imposing that all these coefficients vanish, we determine equations of degree $n+1$ for $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ as a closed subscheme of $\mathbf{Gr}^{p(r)}_{S_r}(k)$. □

Example 4.2. Let us examine again the case of the Hilbert scheme $\mathbf{Hilb}_2^{\mathbb{P}^2}(K)$ with a field extension $k \subset K$ and the same notation defined in Example 3.2. Since $q(2) = 4$ and $q(3) + 1 = 9$, we have to compute the exterior products $x\,\delta_{\mathcal{J}_0}^{m_0} \wedge y\,\delta_{\mathcal{J}_1}^{m_1} \wedge z\,\delta_{\mathcal{J}_2}^{m_2}$ for $(m_0, m_1, m_2) = (4,4,1), (4,1,4), (1,4,4), (4,3,2), (4,2,3), (3,4,2), (3,2,4), (2,4,3), (2,3,4), (3,3,3)$. For instance let us write explicitly the value of $x\,\delta_{123456}^4 \wedge y\,\delta_{1346}^2 \wedge z\,\delta_{23456}^3$:

$$\begin{aligned}
x\,\delta_{123456}^4 = {} & \Delta_{56}\, x^3 \wedge x^2y \wedge xy^2 \wedge x^2z - \Delta_{46}\, x^3 \wedge x^2y \wedge xy^2 \wedge xyz + \Delta_{45}\, x^3 \wedge x^2y \wedge xy^2 \wedge xz^2 + \\
& + \Delta_{36}\, x^3 \wedge x^2y \wedge x^2z \wedge xyz - \Delta_{35}\, x^3 \wedge x^2y \wedge x^2z \wedge xz^2 + \Delta_{34}\, x^3 \wedge x^2y \wedge xyz \wedge xz^2 + \\
& - \Delta_{26}\, x^3 \wedge xy^2 \wedge x^2z \wedge xyz + \Delta_{25}\, x^3 \wedge xy^2 \wedge x^2z \wedge xz^2 - \Delta_{24}\, x^3 \wedge xy^2 \wedge xyz \wedge xz^2 + \\
& + \Delta_{23}\, x^3 \wedge x^2z \wedge xyz \wedge xz^2 + \Delta_{16}\, x^2y \wedge xy^2 \wedge x^2z \wedge xyz - \Delta_{15}\, x^2y \wedge xy^2 \wedge x^2z \wedge xz^2 + \\
& + \Delta_{14}\, x^2y \wedge xy^2 \wedge xyz \wedge xz^2 - \Delta_{13}\, x^2y \wedge x^2z \wedge xyz \wedge xz^2 + \Delta_{12}\, xy^2 \wedge x^2z \wedge xyz \wedge xz^2, \\
y\,\delta_{1346}^2 = {} & \Delta_{46}\, x^2y \wedge y^3 - \Delta_{36}\, x^2y \wedge xyz + \Delta_{34}\, x^2y \wedge yz^2 + \Delta_{16}\, y^3 \wedge xyz - \Delta_{14}\, y^3 \wedge yz^2 + \Delta_{13}\, xyz \wedge yz^2, \\
z\,\delta_{23456}^3 = {} & \Delta_{56}\, xyz \wedge y^2z \wedge xz^2 - \Delta_{46}\, xyz \wedge y^2z \wedge yz^2 + \Delta_{45}\, xyz \wedge y^2z \wedge z^3 + \Delta_{36}\, xyz \wedge xz^2 \wedge yz^2 + \\
& - \Delta_{35}\, xyz \wedge xz^2 \wedge z^3 + \Delta_{34}\, xyz \wedge yz^2 \wedge z^3 - \Delta_{26}\, y^2z \wedge xz^2 \wedge yz^2 + \Delta_{25}\, y^2z \wedge xz^2 \wedge z^3 + \\
& - \Delta_{24}\, y^2z \wedge yz^2 \wedge z^3 + \Delta_{23}\, xz^2 \wedge yz^2 \wedge z^3.
\end{aligned}$$

The coefficients of the exterior product of these three elements are polynomials in the Plücker coordinates of degree 3 that belong to the ideal defining the Hilbert scheme $\mathbf{Hilb}_2^{\mathbb{P}^2}(K)$:

- $-\Delta_{26}^2\Delta_{46} + \Delta_{25}\Delta_{46}^2 + \Delta_{16}\Delta_{26}\Delta_{56} - \Delta_{14}\Delta_{56}^2$,
- $+\Delta_{25}\Delta_{26}\Delta_{46} - \Delta_{25}\Delta_{45}\Delta_{46} - \Delta_{16}\Delta_{25}\Delta_{56}$,
- $-\Delta_{24}\Delta_{26}\Delta_{46} + \Delta_{16}\Delta_{24}\Delta_{56} + \Delta_{14}\Delta_{45}\Delta_{56}$,
- $+\Delta_{23}\Delta_{26}\Delta_{46} + \Delta_{25}\Delta_{34}\Delta_{46} - \Delta_{16}\Delta_{23}\Delta_{56} - \Delta_{14}\Delta_{35}\Delta_{56}$,
- $-\Delta_{24}\Delta_{25}\Delta_{46} + \Delta_{14}\Delta_{25}\Delta_{56}$,
- $+\Delta_{16}\Delta_{24}\Delta_{45} + \Delta_{14}\Delta_{45}^2 + \Delta_{24}^2\Delta_{46} - \Delta_{14}\Delta_{25}\Delta_{46}$,
- $+\Delta_{25}\Delta_{26}\Delta_{34} - \Delta_{24}\Delta_{25}\Delta_{36} - \Delta_{25}\Delta_{34}\Delta_{45} + \Delta_{13}\Delta_{25}\Delta_{56}$,
- $+\Delta_{16}\Delta_{24}\Delta_{35} - \Delta_{14}\Delta_{25}\Delta_{36} + \Delta_{14}\Delta_{35}\Delta_{45} + \Delta_{23}\Delta_{24}\Delta_{46}$,
- $-\Delta_{16}\Delta_{24}\Delta_{25} + \Delta_{14}\Delta_{25}\Delta_{26} - \Delta_{14}\Delta_{25}\Delta_{45}$,
- $+\Delta_{15}\Delta_{16}\Delta_{24} - \Delta_{14}\Delta_{16}\Delta_{25} + \Delta_{14}\Delta_{15}\Delta_{45} - \Delta_{12}\Delta_{24}\Delta_{46}$.

5. BLMR equations

Throughout last two sections, given an admissible Hilbert polynomial $p(t)$ on $\mathbb{P}^n$ with Gotzmann number r and degree d and the associated volume polynomial $q(t) = \binom{n+t}{n} - p(t)$, we set

$$q'(t) = q(t) - \binom{n-d-1+t}{n-d-1} = \binom{n+t}{n} - \binom{n-d-1+t}{n-d-1} - p(t), \tag{11}$$

so that $q(t) - q'(t) = \dim_k k[x_{d+1}, \dots, x_n]_t$.

Proposition 5.1. *Let $\mathcal{U}'$ be the set of all the elements $I_r \in \mathbf{Gr}_{S_r}^{p(r)}(k)$ (i.e. S_r/I_r has dimension $p(r)$) such that I_r has a set of generators of the type:*

$$G_I^r = \left\{x^\alpha + f_\alpha \mid x^\alpha \in k[x_{d+1}, \dots, x_n]_r \text{ and } f_\alpha \in (x_d, \dots, x_0)\right\} \cup \left\{g_j \mid g_j \in (x_d, \dots, x_0)\right\} \tag{12}$$

where the first set of generators contains by construction $q(r) - q'(r)$ elements and the second set has $q'(r)$ polynomials, so that $\dim_k I_r = q(r)$.

Then $\mathcal{U}'$ is a non-empty open subset in $\mathbf{Gr}_{S_r}^{p(r)}(k)$ and I_{r+1} has a set of generators G_I^{r+1} that can be represented by a matrix of the type:

$$\mathcal{A}_{r+1} = \left(\begin{array}{c|c} \mathtt{Id} & \bullet \quad \bullet \quad \bullet \\ \hline 0 & \mathcal{D}_1 \\ \hline 0 & \mathcal{D}_2 \end{array}\right) \tag{13}$$

where:

- *the columns belonging to the left part of the matrix correspond to the monomials in* $k[x_{d+1},\dots,x_n]_{r+1}$ *and the columns on the right to the monomials in* $(x_0,\dots,x_d)_{r+1}$*;*
- *the top-left submatrix* `Id` *is the identity matrix of order* $q(r+1)-q'(r+1)$*;*
- *the rows of* $\mathcal{D}_1$ *contain the coefficients of all the generators multiplied by a variable* $x_h,\ h=0,\dots,d$*;*
- *the rows of* $\mathcal{D}_2$ *contain the coefficients of the generators* g_j *multiplied by a variable* $x_h,\ h=d+1,\dots,n$ *and the coefficients of the polynomials* $x_{i'}f_{\alpha'}-x_if_\alpha$ *such that* $x_{i'}x^{\alpha'}=x_ix^\alpha$ *and* $i,i'\geqslant d+1$*.*

Moreover the subset $\mathcal{U}\subset\mathcal{U}'$ *of all the ideals* I_r *such that* $\operatorname{rank}\mathcal{D}_1\geqslant q'(r+1)$ *is open and* $\mathcal{U}^{\mathrm{PGL}}=\{g\mathcal{U}\mid g\in\mathrm{PGL}(n+1)\}$ *is an open covering of* $\mathbf{Gr}^{p(r)}_{S_r}(k)$*.*

Proof. Let us consider the canonical projection

$$\pi: k[x_0,\dots,x_n]_r \longrightarrow (k[x_0,\dots,x_n]/(x_0,\dots x_d))_r\simeq k[x_{d+1},\dots,x_n]_r.$$

The subset $\mathcal{U}'$ of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ is open because $\mathcal{U}'=\pi^{-1}(k[x_{d+1},\dots,x_n]_r)\cap\mathbf{Gr}^{p(r)}_{S_r}(k)$. Moreover $\mathcal{U}'$ is non-empty because any Borel ideal J defining a point of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ (i.e. $\dim_k J_r=q(r)$) belongs to $\mathcal{U}'$. Indeed, $\dim_k J_t\geqslant q(t),\ \forall\, t\geqslant r$ (by Macaulay's Estimate on the Growth of Ideals) implies that the Hilbert polynomial of $k[x_0,\dots,x_n]/J$ has degree smaller than or equal to $\deg p(t)=d$ and so $k[x_{d+1},\dots,x_n]_{\geqslant r}\subseteq J$ (see [5, Proposition 2.3]). Therefore every ideal $I_r\in\mathcal{U}'$ has a basis G^r_I (of I_r as k-vector space) as the one described in (12).

A set of generators for I_{r+1} is $S_1\cdot G^r_I=\bigcup_i\{x_i\,G^r_I\}$. The set of generators G^{r+1}_I we are looking for can be easily obtained from $S_1\cdot G^r_I$ just modifying few elements: for every monomial x^γ in $k[x_{d+1},\dots,x_n]_{r+1}$ we choose only one product $x_i(x^\alpha+f_\alpha)$ such that $x^\gamma=x_i\,x^\alpha$, to be left in G^{r+1}_I (and corresponding to a row in the first block of $\mathcal{A}_{r+1}$), whereas we replace any other polynomial $x_{i'}(x^{\alpha'}+f_{\alpha'})$, such that $x^\gamma=x_{i'}x^{\alpha'}$, by $x_{i'}f_{\alpha'}-x_if_\alpha$ (which belongs to $(x_0,\dots,x_d)$ and corresponds to a row of $\mathcal{D}_2$).

Obviously the condition $\operatorname{rank}\mathcal{D}_1\geqslant q'(r+1)$ is an open condition and we call $\mathcal{U}\subset\mathcal{U}'$ the corresponding open subset. Again this open subset is not empty because it contains for instance all the subspaces J_r defined by a Borel ideal $J\in\mathbf{Gr}^{p(r)}_{S_r}(k)$.

To prove the last statement, we consider any term ordering $\preceq$ such that $x_n\succ\dots\succ x_0$, and we recall that in general coordinates the initial ideal of any ideal is Borel-fixed (see [6, Theorem 15.20]). Then for a general $g\in\mathrm{PGL}(n+1)$, $J=(\mathrm{in}(gI)_r)$ is Borel. Note that J belongs to $\mathbf{Gr}^{p(r)}_{S_r}(k)$, but if $I\notin\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$, J can differ from $\mathrm{in}(gI)$. As J is Borel-fixed

$$\begin{aligned}\dim_k(x_0J_r+\dots+x_dJ_r)&=\dim_k J_{r+1}\cap(x_0,\dots,x_d)=\\&=\dim_k J_{r+1}-\dim_k k[x_{d+1},\dots,x_n]_{r+1}\geqslant q'(r+1),\end{aligned}$$

hence

$$\begin{aligned}\dim_k\big(x_0(gI)_r+\dots+x_d(gI)_r\big)&\geqslant\dim_k\big(x_0\mathrm{in}(gI)_r+\dots+x_d\mathrm{in}(gI)_r\big)=\\&=\dim_k(x_0J_r+\dots+x_dJ_r)\geqslant q'(r+1).\end{aligned}$$

Finally we can conclude that $gI\in\mathcal{U}$ because a set of generators of the vector space $x_0(gI)_r+\dots+x_d(gI)_r$ corresponds to the rows of $\mathcal{D}_1$. □

Making reference to the above matrix $\mathcal{A}_{r+1}$, let us denote by $\mathcal{D}$ the submatrix $\begin{pmatrix}\mathcal{D}_1\\ \mathcal{D}_2\end{pmatrix}$.

Corollary 5.2. *Let* $I\subset S$ *be any ideal belonging to* $\mathcal{U}$*. Then*

(1) $I\in\mathcal{U}\cap\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)\quad\Longleftrightarrow\quad\operatorname{rank}\mathcal{D}=\operatorname{rank}\mathcal{D}_1=q'(r+1)$*;*

(2) $I\notin\mathcal{U}\cap\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)\quad\Longleftrightarrow\quad$ *either* $\operatorname{rank}\mathcal{D}_1>q'(r+1)$ *or* $\operatorname{rank}\mathcal{D}_1=q'(r+1)<\operatorname{rank}\mathcal{D}$*.*

Proof. *(1)* follows from the fact that $I \in \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ if and only if $\dim_k I_{r+1} = \operatorname{rank} \mathcal{A}_{r+1} = q(r+1)$ and from the special form of $\mathcal{A}_{r+1}$. *(2)* is another way to write *(1)*. □

Theorem 5.3. *Let $p(t)$ be an admissible Hilbert polynomial in $\mathbb{P}^n$ of degree d and Gotzmann number r. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ can be defined as a closed subscheme of the grassmannian $\mathbf{Gr}^{p(r)}_{S_r}(k)$ by equations of degree smaller than or equal to $d+2$.*

Proof. We divide the proof in two steps.
Step 1. First of all we construct a set of polynomials T in $k[\Delta]$ such that given an ideal $I \in \mathcal{U}$, every polynomial in T vanishes on the Plücker coordinates $[\ldots : \Delta(I) : \ldots]$ of I if and only if $I \in \mathcal{U} \cap \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$.

Let us choose in all the possible ways a set of $d+1$ elements of the type $x_i \delta^{m_i}_{\mathcal{J}_i}$ such that $i = 0, \ldots, d$ and $\sum_i m_i = q'(r+1)$. Moreover let us consider any multi-index $\mathcal{K} = \{k_1, \ldots, k_{p(r)+1}\}$, such that the corresponding monomials $\{x^{\alpha(k_1)}, \ldots, x^{\alpha(k_{p(r)+1})}\}$ of degree r do not belong to $k[x_{d+1}, \ldots, x_n]_r$, and any variable x_j, $j = d+1, \ldots, n$. A first part T' of the polynomials in T is represented by all the coefficients of all exterior products of the type

$$\left(\bigwedge_{i=0,\ldots,d} x_i\, \delta^{m_i}_{\mathcal{J}_i} \right) \wedge x_j\, \delta^1_{\mathcal{K}}. \tag{14}$$

Afterwards let us choose in all the possible ways a multi-index $\mathcal{H} = \{h_1, \ldots, h_{p(r)}\}$, such that the corresponding monomials $\{x^{\alpha(h_1)}, \ldots, x^{\alpha(h_{p(r)})}\}$ of degree r do not belong to $k[x_{d+1}, \ldots, x_n]_r$ and a monomial $x^\gamma \in k[x_{d+1}, \ldots, x_n]_{r-1}$ with two indices $\overline{j}, \overline{j}' \in \{d+1, \ldots, n\}$. The second part T'' of the polynomials in T can be obtained collecting the coefficients of all the exterior products of the type

$$\left(\bigwedge_{i=0,\ldots,d} x_i\, \delta^{m_i}_{\mathcal{J}_i} \right) \wedge \left(x_{\overline{j}'}\, \delta^1_{\mathcal{H}_1} - x_{\overline{j}}\, \delta^1_{\mathcal{H}_2} \right) \tag{15}$$

where, called $\overline{h}$ and $\overline{h}'$ the indices such that $x^{\alpha(\overline{h})} = x_{\overline{j}} x^\gamma$ and $x^{\alpha(\overline{h}')} = x_{\overline{j}'} x^\gamma$, $\mathcal{H}_1 = \mathcal{H} \cup \{\overline{h}\} = \{\overline{h}, h_1, \ldots, h_{p(r)}\}$ and $\mathcal{H}_2 = \mathcal{H} \cup \{\overline{h}'\} = \{\overline{h}', h_1, \ldots, h_{p(r)}\}$.

Note that the conditions given by the vanishing of the polynomials in $T = T' \cup T''$ mean that $q'(r+1)$ rows in the matrix $\mathcal{D}_1$ and one rows in the matrix $\mathcal{D}_2$ are linearly dependent (directly by the construction of the matrix $\mathcal{A}_{r+1}$ in Proposition 5.1). These conditions ensure also that $q'(r+1)+1$ rows of the matrix $\mathcal{D}_1$ are dependent, because of the well-known property of vector spaces saying that $s+1$ vectors $v_1, \ldots, v_{s+1}$, such that every subset of s elements is linear dependent with any other vector $u \neq 0$, are dependent. The only delicate issue, that we will discuss later in Remark 5.4, is checking that $\mathcal{D}_2$ is not a zero matrix.

By Corollary 5.2, $I \in \mathcal{U}$ belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ if and only if the polynomials of T vanish on $[\ldots : \Delta(I) : \ldots]$. Note that the coefficients of the exterior products in (14) and (15) are polynomials of degree $\leqslant d+2$ (more precisely the degree is the number of non-zero m_i).

Step 2. Let I be an element of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ and $g = (g_{i,j})$ be an element of $\mathrm{PGL}(n+1)$. The Plücker coordinates $[\ldots : \Delta(gI) : \ldots]$ of $gI \in \mathbf{Gr}^{p(r)}_{S_r}(k)$ are bi-homogeneous polynomials of degree 1 in the Plücker coordinates $[\ldots : \Delta(I) : \ldots]$ and of degree $q(r) \cdot r$ in the coefficients $g_{i,j}$ of the matrix g. So given a homogeneous polynomial P of degree $s \leqslant d+2$ in T, $P([\ldots : \Delta(gI) : \ldots])$ is a bi-homogeneous polynomial of degree s in $[\ldots : \Delta(I) : \ldots]$ and of degree $q(r) \cdot r \cdot s$ in $g_{i,j}$. At this point we collect, and denote by C_P, the homogeneous polynomials of degree $s \leqslant d+2$ in the Plücker coordinates $[\ldots : \Delta(I) : \ldots]$, that spring up as coefficients of $P([\ldots : \Delta(gI) : \ldots])$, viewed as a homogeneous polynomial of degree $q(r) \cdot r \cdot s$ in the variables $g_{i,j}$.

From Proposition 5.1 and Corollary 5.2, I belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ if and only if for a generic changes of variables $g \in \mathrm{PGL}(n+1)$

$$gI \in \mathcal{U} \cap \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$$

i.e.

$$\text{all the homogeneous polynomials } P \in T \text{ vanish at } [\ldots : \Delta(gI) : \ldots],$$

or equivalently

$$\text{all the coefficients } C_P \text{ for } P \in T \text{ vanish at } [\ldots : \Delta(I) : \ldots].$$

We finally proved that $I \in \mathbf{Gr}^{p(r)}_{S_r}(k)$ belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ if and only if $[\ldots : \Delta(I) : \ldots]$ satisfies all the equations of the set

$$\bigcup_{P \in T} C_P \tag{16}$$

which consists of homogeneous polynomial of degree smaller than or equal to $d+2$. $\square$

Remark 5.4. Note that a necessary condition in order that $\mathcal{D}_2$ is empty is that I_r has no generators belonging to G_I^r of the type g_j and now we prove that it is not possible. For the sake of simplicity, we can think about the monomial ideal obtained in the case $f_\alpha = g_j = 0$, $\forall\ \alpha$, $\forall\ j$. Such an ideal should have a Hilbert polynomial $\widetilde{p}(t)$ such that $\widetilde{p}(r) = \binom{n+r}{n} - \binom{n-d+r}{n-d}$. Let us show that $\widetilde{p}(r)$ can not be equal to $P(r)$ with P a Hilbert polynomial with Gotzmann number equal to r and of degree d.

The first point is to compute the maximal value in degree r of a Hilbert polynomial of degree d and Gotzmann number r. By the decomposition of Hilbert polynomials given by Gotzmann's Regularity Theorem [8, Theorem 3.11], we know that among the Hilbert polynomials of degree d and Gotzmann number r there is

$$p(t) = \binom{t+d}{d} + \binom{t+d-1}{d} + \ldots + \binom{t+d-(r-1)}{d},$$

and that any other Hilbert polynomial has at least one binomial coefficient $\binom{t+(d-i)-j}{d-i}$ replacing $\binom{t+d-j}{d}$ $(i,j>0)$. Because of $\binom{r+d-j}{d} > \binom{r+(d-i)-j}{d-i}$, the maximal value reached is

$$P = \max\left\{P(r) \mid P(t) \text{ of degree } d \text{ and Gotzmann number } r\right\} = \sum_{i=1}^{r} \binom{d+i}{d}.$$

Finally, starting from the decomposition

$$k[x_0,\ldots,x_n]_r = \bigcup_{i=0,\ldots,r} k[x_0,\ldots,x_d]_i \cdot k[x_{d+1},\ldots,x_n]_{r-i},$$

we have that

$$\widetilde{p}(r) = \binom{n+r}{n} - \binom{n-d+r+1}{n-d+1} = \sum_{i=1}^{r} \binom{d+i}{d}\binom{n-d+1+r-i}{n-d+1} > \sum_{i=1}^{r}\binom{d+i}{d} = P.$$

Example 5.5. Let us apply Theorem 5.3 to the case already considered in Example 3.2 and Example 4.2. Since the Hilbert polynomial is constant, to compute the first set of equations T' (14), we have to consider the wedge product between $z\delta^4_{123456}$ and the 2 elements $x\delta^1_{456}, y\delta^1_{456}$ not containing monomials in $k[x,y]$. We obtain 12 polynomials:

- $-\Delta_{26}\Delta_{46} + \Delta_{45}\Delta_{46} + \Delta_{16}\Delta_{56}$,
- $-\Delta_{24}\Delta_{46} + \Delta_{14}\Delta_{56}$,
- $-\Delta_{24}\Delta_{45} + \Delta_{12}\Delta_{56}$,
- $-\Delta_{36}\Delta_{46} + \Delta_{26}\Delta_{56} + \Delta_{45}\Delta_{56}$,
- $-\Delta_{45}^2 - \Delta_{34}\Delta_{46} + \Delta_{24}\Delta_{56}$,
- $-\Delta_{25}\Delta_{45} + \Delta_{23}\Delta_{46}$,
- $-\Delta_{45}^2 + \Delta_{25}\Delta_{46} - \Delta_{15}\Delta_{56}$,
- $\Delta_{34}\Delta_{45} + \Delta_{23}\Delta_{46} - \Delta_{13}\Delta_{56}$,
- $\Delta_{14}\Delta_{45} - \Delta_{12}\Delta_{46}$,
- $\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56}$,
- $\Delta_{35}\Delta_{45} - \Delta_{23}\Delta_{56}$,
- $\Delta_{15}\Delta_{45} - \Delta_{13}\Delta_{46} + \Delta_{12}\Delta_{56}$.

To compute the second set of equations T'' (15), we have to consider the coefficients of the wedge product between $z\delta^4_{123456}$ and one element belonging to the set

$$\{y\delta^1_{145} - x\delta^1_{245},\, y\delta^1_{245} - x\delta^1_{345},\, y\delta^1_{146} - x\delta^1_{246},\, y\delta^1_{246} - x\delta^1_{346},\, y\delta^1_{156} - x\delta^1_{256},\, y\delta^1_{256} - x\delta^1_{356}\}.$$

We obtain other 36 generators:

- $\Delta_{16}\Delta_{25} - \Delta_{15}\Delta_{26} - \Delta_{24}\Delta_{26} + \Delta_{14}\Delta_{36}$,
- $-\Delta_{15}\Delta_{24} - \Delta_{24}^2 + \Delta_{14}\Delta_{25} + \Delta_{14}\Delta_{34}$,
- $-\Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{25}$,
- $-\Delta_{25}\Delta_{26} - \Delta_{26}\Delta_{34} + \Delta_{16}\Delta_{35} + \Delta_{24}\Delta_{36}$,
- $-\Delta_{24}\Delta_{25} + \Delta_{14}\Delta_{35}$,
- $-\Delta_{23}\Delta_{24} + \Delta_{12}\Delta_{35}$,
- $-\Delta_{24}\Delta_{46} + \Delta_{14}\Delta_{56}$,
- $-\Delta_{16}\Delta_{24} + \Delta_{14}\Delta_{26} - \Delta_{14}\Delta_{45}$,
- $\Delta_{24}^2 - \Delta_{14}\Delta_{25} + \Delta_{12}\Delta_{26}$,
- $-\Delta_{26}^2 + \Delta_{16}\Delta_{36} - \Delta_{34}\Delta_{46} + \Delta_{24}\Delta_{56}$,
- $-\Delta_{24}\Delta_{26} + \Delta_{14}\Delta_{36} - \Delta_{24}\Delta_{45}$,
- $-\Delta_{24}\Delta_{25} + \Delta_{24}\Delta_{34} + \Delta_{12}\Delta_{36}$,
- $\Delta_{26}^2 - \Delta_{16}\Delta_{36} - \Delta_{25}\Delta_{46} + \Delta_{15}\Delta_{56}$,
- $\Delta_{24}\Delta_{26} - \Delta_{16}\Delta_{34} - \Delta_{15}\Delta_{45}$,
- $\Delta_{16}\Delta_{23} - \Delta_{15}\Delta_{25} + \Delta_{24}\Delta_{25}$,
- $-\Delta_{35}\Delta_{46} + \Delta_{25}\Delta_{56}$,
- $-\Delta_{26}\Delta_{34} + \Delta_{24}\Delta_{36} - \Delta_{25}\Delta_{45}$,
- $-\Delta_{25}^2 + \Delta_{23}\Delta_{26} + \Delta_{24}\Delta_{35}$,
- $\Delta_{24}\Delta_{25} - \Delta_{14}\Delta_{35}$,
- $\Delta_{15}\Delta_{23} + \Delta_{23}\Delta_{24} - \Delta_{13}\Delta_{25}$,
- $\Delta_{13}\Delta_{14} - \Delta_{12}\Delta_{15} - \Delta_{12}\Delta_{24}$,
- $\Delta_{25}^2 + \Delta_{25}\Delta_{34} - \Delta_{15}\Delta_{35} - \Delta_{24}\Delta_{35}$,
- $\Delta_{23}\Delta_{25} + \Delta_{23}\Delta_{34} - \Delta_{13}\Delta_{35}$,
- $\Delta_{13}\Delta_{24} - \Delta_{12}\Delta_{25} - \Delta_{12}\Delta_{34}$,
- $\Delta_{16}\Delta_{25} - \Delta_{15}\Delta_{26} + \Delta_{24}\Delta_{45}$,
- $\Delta_{16}\Delta_{23} - \Delta_{13}\Delta_{26} - \Delta_{24}\Delta_{34} + \Delta_{14}\Delta_{35}$,
- $\Delta_{14}\Delta_{15} - \Delta_{12}\Delta_{16} - \Delta_{14}\Delta_{24}$,
- $\Delta_{25}\Delta_{26} - \Delta_{15}\Delta_{36} + \Delta_{34}\Delta_{45}$,
- $\Delta_{23}\Delta_{26} - \Delta_{34}^2 + \Delta_{24}\Delta_{35} - \Delta_{13}\Delta_{36}$,
- $\Delta_{15}\Delta_{24} - \Delta_{12}\Delta_{26} - \Delta_{14}\Delta_{34}$,
- $-\Delta_{25}\Delta_{26} + \Delta_{16}\Delta_{35} + \Delta_{25}\Delta_{45}$,
- $-\Delta_{23}\Delta_{26} - \Delta_{25}\Delta_{34} + \Delta_{15}\Delta_{35}$,
- $\Delta_{15}^2 - \Delta_{13}\Delta_{16} - \Delta_{14}\Delta_{25} + \Delta_{12}\Delta_{26}$,
- $\Delta_{26}\Delta_{35} - \Delta_{25}\Delta_{36} + \Delta_{35}\Delta_{45}$,
- $\Delta_{25}\Delta_{35} - \Delta_{34}\Delta_{35} - \Delta_{23}\Delta_{36}$,
- $\Delta_{15}\Delta_{25} - \Delta_{13}\Delta_{26} - \Delta_{14}\Delta_{35} + \Delta_{12}\Delta_{36}$.

Now we need to introduce the action of $\mathrm{PGL}(3)$ on $k[x,y,z]$ and to understand how the induced action on $\mathbf{Gr}^4_{k[x,y,z]_2}(K)$ works. Given an element $g = (g_{i,j}) \in \mathrm{PGL}(3)$ and its action

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longleftarrow \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the induced action on the k-vector space $k[x,y,z]_2$ is represented by the matrix

$$\begin{pmatrix} x^2 \\ xy \\ y^2 \\ xz \\ yz \\ z^2 \end{pmatrix} \longleftarrow \begin{pmatrix} g_{11}^2 & 2g_{11}g_{12} & g_{12}^2 & 2g_{11}g_{13} & 2g_{12}g_{13} & g_{13}^2 \\ g_{11}g_{21} & g_{12}g_{21}+g_{11}g_{22} & g_{12}g_{22} & g_{13}g_{21}+g_{11}g_{23} & g_{13}g_{22}+g_{12}g_{23} & g_{13}g_{23} \\ g_{21}^2 & 2g_{21}g_{22} & g_{22}^2 & 2g_{21}g_{23} & 2g_{22}g_{23} & g_{23}^2 \\ g_{11}g_{31} & g_{12}g_{31}+g_{11}g_{32} & g_{12}g_{32} & g_{13}g_{31}+g_{11}g_{33} & g_{13}g_{32}+g_{12}g_{33} & g_{13}g_{33} \\ g_{21}g_{31} & g_{22}g_{31}+g_{21}g_{32} & g_{22}g_{32} & g_{23}g_{31}+g_{21}g_{33} & g_{23}g_{32}+g_{22}g_{33} & g_{23}g_{33} \\ g_{31}^2 & 2g_{31}g_{32} & g_{32}^2 & 2g_{31}g_{33} & 2g_{32}g_{33} & g_{33}^2 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ y^2 \\ xz \\ yz \\ z^2 \end{pmatrix}.$$

To write explicitly the action of g on the Plücker coordinates, we can consider the element δ^4_{123456}, substitute each element x^β of the basis of $k[x,y,z]_2$ with $g.x^\beta$ obtaining the exterior product $g.\delta^4_{123456}$ and then the action is determined by looking at the coefficients of the same element of the basis of $\wedge^4 k[x,y,z]_2$ in δ^4_{123456} and $g.\delta^4_{123456}$. For instance the Plücker coordinate Δ_{25}, coefficient of $x^2 \wedge y^2 \wedge xz \wedge z^2$ in δ^4_{123456}, will be send by the action of g to the coefficient of $x^2 \wedge y^2 \wedge xz \wedge z^2$ in $g.\delta^4_{123456}$.

Finally, for every polynomial P contained in T, we have to compute the action of g, that is substituting each Δ_{ab} with $g.\Delta_{ab}$, and then we have to collect all the coefficients (polynomials in the Plücker coordinates of degree 2) of $g.P$, viewed as polynomial in the variables $g_{i,j}$. For instance collecting the coefficients of the polynomial $g.(\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56})$, $\Delta_{35}\Delta_{46} - \Delta_{25}\Delta_{56} \in T'$, we obtain 3495 polynomials that give some of the equations defining the Hilbert scheme $\mathbf{Hilb}_2^{\mathbb{P}^2}(K)$.

6. Generalization

Let us prove now that equations obtained from Theorem 5.3 can be used to define the Hilbert scheme, as scheme representing the Hilbert functor (see for instance [16, Section 4.3]).

Let us start recalling some relevant definitions and properties.

Definition 6.1 ([11])**.** Let X and Y be schemes and $f : X \to Y$ be a morphism of schemes. X is said to be flat over Y if $\mathcal{O}_X$ is f-flat over Y i.e. for every $x \in X$, $O_{X,x}$ is a $O_{Y,f(x)}$-flat module.

Definition 6.2 ([16])**.** The Hilbert functor of $\mathbb{P}^n$ relative to the polynomial $p(t)$ is the contravariant functor $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}$: (schemes)$^\circ \to$ Sets from the category of (schemes) to the category of Sets which associates to an object X of (schemes) the set

$$\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X) = \left\{ \begin{array}{c} \text{flat families } \mathcal{Z} \subset \mathbb{P}^n \times X \text{ of closed} \\ \text{subschemes of } \mathbb{P}^n \text{ parametrized by } X \text{ with} \\ \text{fibers having Hilbert polynomial } p(t) \end{array} \right\} \tag{17}$$

The Hilbert functor is representable and we will denote by $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ its representing scheme.

Example 6.3. If $X = \operatorname{Spec} A$, with A noetherian k-algebra, $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(A) = \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(\operatorname{Spec} A)$ is given by the set of saturated homogeneous ideals $I \subset A[x_0,\dots,x_n]$ such that $\operatorname{Proj} A[x_0,\dots,x_n]/I$ is flat over $\operatorname{Spec} A$ and for every prime ideal $\mathfrak{p} \subset A$, the Hilbert polynomial of the $k(\mathfrak{p})$-graded algebra $A[x_0,\dots,x_n]/I \otimes_A k(\mathfrak{p})$ is equal to $p(t)$ where $k(\mathfrak{p})$ is the residue field $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Proposition 6.4. *Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring such that R_0 is a local noetherian ring and that R is finitely generated as R_0-algebra by elements of degree 1. Let M be a finitely generated graded R-module and let M_t be its component of degree t. Then, M is flat over R_0 if and only if M_t is free over R_0 for all t.*

Proof. See [6, Exercise 6.10]. □

Corollary 6.5. *Let $X = \operatorname{Spec} A$, where A is a noetherian k-algebra and a local ring.* $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(A)$ *is equal to the set of homogeneous saturated ideals I of $A[x_0,\dots,x_n]$ such that $A[x_0,\dots,x_n]_t/I_t$ is a free A-module for all $t \geqslant 0$ and such that the rank of $A[x_0,\dots,x_n]_t/I_t$ is equal to $p(t)$ for $t \gg 0$.*

We recall two theorems by Gotzmann already introduced, but now stated in the more general case of coefficients in a local ring (see [7, 12]).

Theorem 6.6 (Gotzmann's Regularity Theorem)**.** *Let r be the Gotzmann number associated to the Hilbert polynomial $p(t)$, A any noetherian ring and $T = A[x_0,\dots,x_n]$. Any homogeneous ideal $I \subset T$ with Hilbert polynomial $p(t)$ is r-regular, that is*

$$H^i(\mathbb{P}^n, \widetilde{I}(r-i)) = 0$$

for $i > 0$, where $\widetilde{I}$ is the quasi-coherent sheaf associated to I.

Theorem 6.7 (Gotzmann's Persistence Theorem)**.** *Let r be the Gotzmann number associated to the Hilbert polynomial $p(t)$ and $s \geqslant r$ an integer. Let A be any noetherian ring, $T = A[x_0,\dots,x_n]$ and I a homogeneous ideal of T generated by I_s. Set $M = T/I$, if M_t is a flat A-module of rank $p(t)$ for $t = s, s+1$, then M_t is flat of rank $p(t)$ for all $t \geqslant r$.*

Remark 6.8. By Theorem 6.6, we deduce that if I is a saturated homogeneous ideal of T with Hilbert polynomial $p(t)$ whose Gotzmann number is r, then:

(i) $I_{\geqslant t} = (I_t)$, $\forall\, t \geqslant r$;
(ii) the value of Hilbert function of T/I coincides with the value of $p(t)$ for each degree greater than or equal to r.

Now we focus our attention on the case of an affine scheme $X = \operatorname{Spec} A$, with A a noetherian k-algebra and a local ring with maximal ideal $\mathfrak{m}$ and residue field K. Moreover we fix a Hilbert polynomial $p(t)$ of degree d and Gotzmann number r.

Applying Theorems 6.6 and 6.7, Remark 6.8, Corollary 6.5 and Nakayama's Lemma, we deduce the following propositions.

Proposition 6.9. *Given an integer $s \geqslant r$, $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$ is in bijection with the subset W of $\mathbf{Gr}^{p(s)}_{S_s}(X) \times \mathbf{Gr}^{p(s+1)}_{S_{s+1}}(X)$ defined by*

$$W = \left\{ (T_s/I_s, T_{s+1}/J_{s+1}) \in \mathbf{Gr}^{p(s)}_{S_s}(X) \times \mathbf{Gr}^{p(s+1)}_{S_{s+1}}(X) \mid T_1 \cdot I_s = J_{s+1} \right\}. \tag{18}$$

Remark 6.10. Keeping in mind Macaulay's Estimate on the Growth of Ideals (see also [12, Corollary C.4]) and Nakayama's Lemma, the subset W of $\mathbf{Gr}^{p(s)}_{S_s}(X) \times \mathbf{Gr}^{p(s+1)}_{S_{s+1}}(X)$ is also equal to

$$W = \left\{ (T_s/I_s, T_{s+1}/J_{s+1}) \in \mathbf{Gr}^{p(s)}_{S_s}(X) \times \mathbf{Gr}^{p(s+1)}_{S_{s+1}}(X) \mid T_1 \cdot I_s \subseteq J_{s+1} \right\}. \tag{19}$$

Proposition 6.11 ([12]). *Given an integer $s \geqslant r$, $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$ is in bijection with the subset G of $\mathbf{Gr}^{p(s)}_{S_s}(X)$ given by*

$$G = \left\{ T_s/I_s \in \mathbf{Gr}^{p(s)}_{S_s}(X) \mid T_{s+1}/(T_1 \cdot I_s) \textit{ is a free } A\textit{-module of rank } p(s+1) \right\}. \tag{20}$$

Proposition 6.11 says that $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$ can be viewed as a subset of $\mathbf{Gr}^{p(s)}_{S_s}(X)$ for $s \geqslant r$. Now we construct explicitly a set of equations that determine the Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ representing the functor $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}$ as subscheme of the scheme $\mathrm{Gr}^{p(r)}(S_r)$ representing the Grassmann functor $\mathbf{Gr}^{p(r)}_{S_r}$.

Theorem 6.12. *The Hilbert scheme $\mathcal{H}\mathrm{ilb}^n_{p(t)}$ is a closed subscheme of the scheme $\mathrm{Gr}^{p(r)}(S_r)$ representing the Grassmann functor $\mathbf{Gr}^{p(r)}_{S_r}$ and it can be defined by equations in the Plücker coordinates of degree smaller than or equal to $d+2$.*

Proof. Theorem 5.3 proves the statement in the case $X = \operatorname{Spec} k$, that is $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(k)$ is a closed subscheme of $\mathbf{Gr}^{p(r)}_{S_r}(k)$ defined by equations of degree smaller than or equal to $d+2$. Let us extend this result to the case of $X = \operatorname{Spec} A$ with A a noetherian k-algebra and a local ring with maximal ideal $\mathfrak{m}$ and residue field K.

Step 1. Let $I_r \in \mathbf{Gr}^{p(r)}_{S_r}(X)$ be an A-submodule of $T_r = S_r \otimes_k A$, that is T_r/I_r is a free A-module of rank $p(r)$. Firstly let us prove that if equations given in Theorem 5.3 are satisfied, then I_r belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$ (i.e. $T_1 \cdot I_r$ belongs to $\mathbf{Gr}^{p(r+1)}_{S_{r+1}}(X)$ according to Proposition 6.11).

Let us consider $I_r \in \mathbf{Gr}^{p(r)}_{S_r}(X)$ satisfying equations of Theorem 5.3. Tensoring by the residue field K and using Nakayama's Lemma, we can determine a free submodule $J_r \subset T_r$ generated by $q(r)$ monomials having the Borel-fixed property, such that maybe with a generic change of coordinates, the monomials $\mathcal{N}(J_r) = \{x^\beta \in T_r \mid x^\beta \notin J_r\}$ form a basis of T_r/I_r as a free A-module (see [6]). Now we consider the exact sequence

$$0 \to I_r \to T_r \to T_r/I_r \to 0,$$

and we tensor it by the residue field K, obtaining

$$I_r \otimes_k K \to S_r(K) = K[x_0, \dots, x_n]_r \to T_r/I_r \otimes_k K \to 0.$$

Called $I_r(K)$ the image of $I_r \otimes_k K$ in $S_r(K)$, by the assumptions and by Theorem 5.3 we deduce that $I_r(K)$ belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(K)$. Consequently, $J_r(K)$ (resp. J_r) also belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(K)$ (resp. $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(A)$) and thus $(J_r(K))$ (resp. (J_r)) defines a Borel-fixed ideal with Hilbert polynomial $p(t)$ in $S(K) = K[x_0, \dots, x_n]$ (resp. in $A[x_0, \dots, x_n]$).

For a generic change of coordinates, as $\mathcal{N}(J_r)$ is a basis of T_r/I_r, I_r is also a free A-module of rank $q(r)$ with a basis of the form:

$$\left\{ x^\alpha - \sum_{x^\beta \in \mathcal{N}(J_r)} c_{\alpha\beta} x^\beta \;\middle|\; x^\alpha \in J_r \right\}. \tag{21}$$

Therefore we can choose a system of generators for $T_1 \cdot I_r$ equal to that one described with the matrix $\mathcal{A}_{r+1}$ in Proposition 5.1. Up to a change of coordinates, the A-module $\langle \mathcal{D}_1 \rangle$ generated by the lines of $\mathcal{D}_1$ (and by definition equal to $x_0 I_r + \ldots + x_d I_r$) contains a family $\mathcal{F}$ of $q'(r+1)$ polynomials of the form:

$$\mathcal{F} = \left\{ x^\gamma - \sum_{x^\eta \in \mathcal{N}(J_{r+1})} c_{\gamma\eta} x^\eta \;\middle|\; x^\gamma \in x_0 J_r + \ldots + x_d J_r \right\}. \tag{22}$$

Let us prove it by induction on $0 \leqslant i \leqslant d$ for $x^\alpha \in x_i J_r$. Let $x^\gamma = x_0 x^\alpha$ with x^α in J_r. Among the generators (21) of I_r there is

$$x^\alpha - \sum_{x^\beta \in \mathcal{N}(J_r)} c_{\alpha\beta} x^\beta,$$

thus

$$x_0 x^\alpha - \sum_{x^\beta \in \mathcal{N}(J_r)} c_{\alpha\beta} x_0 x^\beta$$

belongs to $x_0 I_r \subset x_0 I_r + \ldots + x_d I_r$. Moreover because of (J_r) is Borel-fixed and $(J_{r+1} : S_1) = J_r$ ($J_r \in \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(A)$), it is easy to check that $x_0 \mathcal{N}(J_r) \subset \mathcal{N}(J_{r+1})$. Hence the assertion is proved for $i = 0$ and let us suppose that it holds for all $0 \leqslant j < i$. Considered $x^\gamma = x_i x^\alpha$ with $x^\alpha \in J_r$, again

$$x_i x^\alpha - \sum_{x^\beta \in \mathcal{N}(J_r)} c_{\alpha\beta} x_i x^\beta \in x_i I_r \subset x_0 I_r + \ldots + x_d I_r.$$

If $x_i x^\beta$ ($x^\beta \in \mathcal{N}(J_r)$) does not belong to $\mathcal{N}(J_{r+1})$, then there exists $x^\epsilon \in J_r$ such that

$$x_i x^\beta = x_j x^\epsilon$$

and $j < i$ because of the Borel-fixed property. Then, by induction, we can replace $x_i x^\beta = x_j x^\epsilon$ with an element of the A-module generated by $\mathcal{N}(J_{r+1})$ modulo $x_0 I_r + \ldots + x_d I_r$, finally proving that the family $\mathcal{F}$ described in (22) belongs to $\langle \mathcal{D}_1 \rangle$ (i.e. to $x_0 I_r + \ldots + x_d I_r$).

As equations of Theorem 5.3 are satisfied, equations (14) and (15) are also satisfied for a generic change of coordinates, so that we can assume without loss of generality that there exist J_r Borel-fixed and $\mathcal{F}$ as in (22), such that I_r satisfies (14) and (15).

Now we want to show that equations (14) and (15) imply that $\mathcal{F}$ generates the A-module $\langle \mathcal{D}_2 \rangle$ spanned by the lines of $\mathcal{D}_2$. As a matter of fact, equations (14) and (15) imply that the exterior product between $q'(r+1)$ polynomials in $\langle \mathcal{D}_1 \rangle$ and one polynomial in $\langle \mathcal{D}_2 \rangle$ always vanishes. In particular, the exterior product between the $q'(r+1)$ polynomials that belong to $\mathcal{F}$ and any polynomial g in $\langle \mathcal{D}_2 \rangle$ is equal to zero. We deduce easily that g belongs to $\langle \mathcal{F} \rangle$ and that $\mathcal{F}$ generates $\langle \mathcal{D}_2 \rangle$.

Moreover $\mathcal{F}$ generates $\langle \mathcal{D}_1 \rangle$. With the same reasoning used in the proof of Theorem 5.3 and in Remark 5.4, it is easy to prove that any exterior product between $q'(r+1)+1$ polynomials in $\langle \mathcal{D}_1 \rangle$ is equal to zero. In particular, the exterior product between the $q'(r+1)$ polynomials that belong to $\mathcal{F}$ and any polynomial g in $\langle \mathcal{D}_1 \rangle$ is equal to zero. So again g belongs to $\langle \mathcal{F} \rangle$ and $\mathcal{F}$ generates $\langle \mathcal{D}_1 \rangle$ (keeping in mind that the free A-module T_r has a basis that contains $\mathcal{F}$).

Finally, we conclude that I_{r+1} is a free A-module with basis $\mathcal{F}$ plus the polynomials represented by the lines in the first rows of $\mathcal{A}_{r+1}$ and rewriting this family of polynomials using linear combinations of elements in $\mathcal{F}$ we can obtain a basis of the form

$$\left\{ x^\gamma - \sum_{x^\eta \in \mathcal{N}(J_{r+1})} c_{\gamma\eta} x^\eta \ \middle|\ x^\gamma \in J_{r+1} \right\}. \tag{23}$$

T_{r+1}/I_{r+1} turns out to be an A-module with basis $\mathcal{N}(J_{r+1})$, so $I_r \in \mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$.

Step 2. Let us suppose that $I_r \in \mathbf{Gr}^{p(r)}_{S_r}(X)$ belongs to $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X)$ and let us prove that it satisfies equations given in Theorem 5.3. This is equivalent to prove that equations (14) and (15) are satisfied for a generic changes of coordinates. From Galligo's Theorem [6, Theorem 15.20] and Nakayama's Lemma, there exists a Borel-fixed monomial ideal J with Hilbert polynomial $p(t)$, such that for a generic changes of coordinates, $\mathcal{N}(J_r)$ and $\mathcal{N}(J_{r+1})$ are a basis of respectively T_r/I_r and T_{r+1}/I_{r+1} as free A-modules. As mentioned in **Step 1**, we can represent I_{r+1} with the matrix $\mathcal{A}_{r+1}$ introduced in Proposition 5.1 and find a family $\mathcal{F}$ in $\langle \mathcal{D}_1 \rangle$ of the form (22).

As $\mathcal{N}(J_{r+1})$ is a basis of T_{r+1}/I_{r+1} as a free A-module, every polynomial given by a line in $\mathcal{D}_1$ or $\mathcal{D}_2$ belongs to $\langle \mathcal{F} \rangle$. Therefore equations (14) and (15) are satisfied for a generic change of coordinates and equations of Theorem 5.3 are satisfied.

Step 3. From the local approach given by Proposition 6.9, the following isomorphism holds

$$\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X) \simeq \left\{ \begin{array}{c} 0 \to \mathcal{I}_r \to S_r \otimes_k \mathcal{O}_X \to \mathcal{E} \to 0 \in \mathbf{Gr}^{p(r)}_{S_r}(X) \\ \text{such that} \\ 0 \to S_1 \cdot \mathcal{I}_r \to S_{r+1} \otimes_k \mathcal{O}_X \to \mathcal{E}' \to 0 \in \mathbf{Gr}^{p(r+1)}_{S_{r+1}}(X) \end{array} \right\} \tag{24}$$

which induces a natural transformation from the Hilbert functor $\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}$ to the Grassmann functor $\mathbf{Gr}^{p(r)}_{S_r}$. Using the equivalence proved in **Step 1** and **Step 2** in a local approach, we deduce

$$\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X) \simeq \left\{ \begin{array}{c} S_r \otimes_k \mathcal{O}_X \to \mathcal{E} \to 0 \in \mathbf{Gr}^{p(r)}_{S_r}(X) \\ \text{such that} \\ \{\Delta_{\mathcal{I}}\}_{\mathcal{I}=\{i_1,\ldots,i_{p(r)}\}} \in H^0(X, \wedge^{p(r)}\mathcal{E}) \text{ satisfy equations of Theorem 5.3} \end{array} \right\} \tag{25}$$

where the element $\Delta_{\mathcal{I}} \in H^0(X, \wedge^{p(r)}\mathcal{E})$ associated to the multi-index $\mathcal{I} = \{i_1, \ldots, i_{p(r)}\}$ is given by the image of $x^{\alpha(i_1)} \wedge \cdots \wedge x^{\alpha(i_{p(r)})} \in H^0(X, \wedge^{p(r)} S_r \otimes_k \mathcal{O}_X)$ into $H^0(X, \wedge^{p(r)}\mathcal{E})$ using the $p(r)$-th exterior power of the morphism:

$$S_r \otimes_k \mathcal{O}_X \ \to\ \mathcal{E}\ \to\ 0.$$

Using the representation of the Grassmann functor $\mathbf{Gr}^{p(r)}_{S_r}$ via the Plücker's relations, we have

$$\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X) \simeq \left\{ \begin{array}{c} S_r \otimes_k \mathcal{O}_X \to \mathcal{E} \to 0 \quad \text{such that} \\ \wedge^{p(r)}\mathcal{E} \text{ is an invertible sheaf and } \{\Delta_{\mathcal{I}}\}_{\mathcal{I}=\{i_1,\ldots,i_{p(r)}\}} \in H^0(X, \wedge^{p(r)}\mathcal{E}) \\ \text{satisfy equations of Theorem 5.3 and Plücker's relations} \end{array} \right\} \tag{26}$$

Finally, because of the isomorphism

$$\mathbf{Hilb}^{\mathbb{P}^n}_{p(t)}(X) \simeq \operatorname{Hom}\left(X, \operatorname{Proj} k[\wedge^{p(r)} S_r]/H\right),$$

(see [11, Theorem 7.1]) where H is the homogeneous ideal of $k[\wedge^{p(r)} S_r]$ generated by Plücker's relations and equations of Theorem 5.3, by definition, the Hilbert scheme is equal to $\operatorname{Proj} k[\wedge^{p(r)} S_r]/H$. $\square$

References

1. Mariemi Alonso, Jerome Brachat, and Bernard Mourrain, *The Hilbert scheme of points and its link with border basis*, available at http://arxiv.org/abs/0911.3503, 2009, preprint.
2. Marilena Barnabei, Andrea Brini, and Gian-Carlo Rota, *On the exterior calculus of invariant theory*, J. Algebra **96** (1985), no. 1, 120–160. MR 808845 (87j:05002)
3. David Bayer, *The division algorithm and the Hilbert schemes*, Ph.D. thesis, Harvard University, 1982.
4. Cristina Bertone, Paolo Lella, and Margherita Roggero, *Borel open covering of Hilbert schemes*, in preparation, 2011.
5. Francesca Cioffi, Paolo Lella, Maria Grazia Marinari, and Margherita Roggero, *Segments and Hilbert schemes of points*, available at http://arxiv.org/abs/1003.2951, 2010, preprint.
6. David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)
7. Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. **158** (1978), no. 1, 61–70. MR 0480478 (58 #641)
8. Mark L. Green, *Generic initial ideals*, Six lectures on commutative algebra (Bellaterra, 1996), Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 119–186. MR MR1648665 (99m:13040)
9. Alexander Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6, Soc. Math. France, Paris, 1995, pp. Exp. No. 221, 249–276. MR 1611822
10. Mark Haiman and Bernd Sturmfels, *Multigraded Hilbert schemes*, J. Algebraic Geom. **13** (2004), no. 4, 725–769. MR MR2073194 (2005d:14006)
11. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157 (57 #3116)
12. Anthony Iarrobino and Vassil Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman. MR MR1735271 (2001d:14056)
13. Paolo Lella, *A network of rational curves on the Hilbert scheme*, available at http://arxiv.org/abs/1006.5020, 2010, preprint.
14. Paolo Lella and Margherita Roggero, *Rational components of Hilbert schemes*, available at http://arxiv.org/abs/0903.1029, 2009, preprint.
15. Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005. MR MR2110098 (2006d:13001)
16. Edoardo Sernesi, *Deformations of algebraic schemes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 334, Springer-Verlag, Berlin, 2006. MR 2247603 (2008e:14011)

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